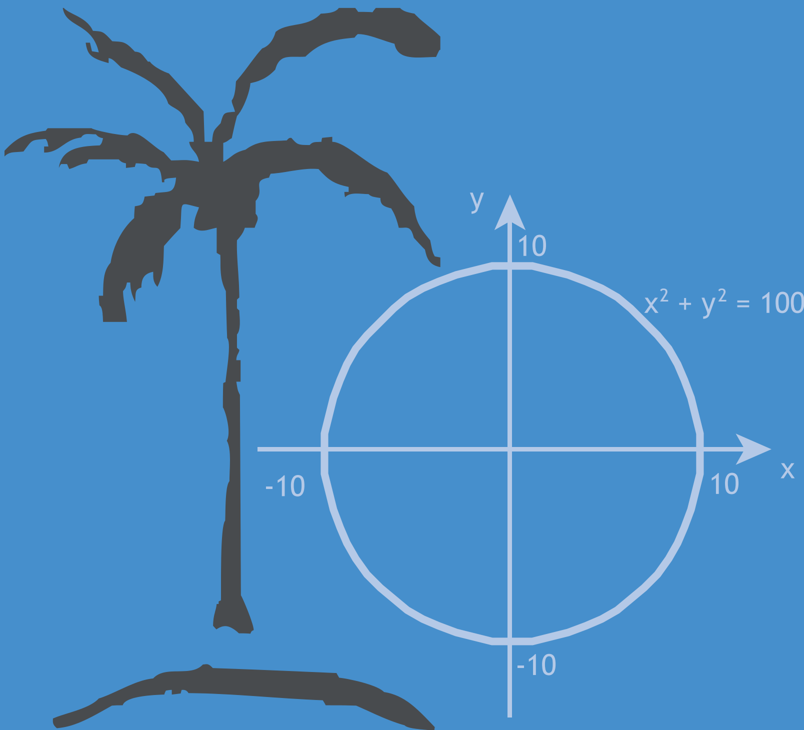
Year 13

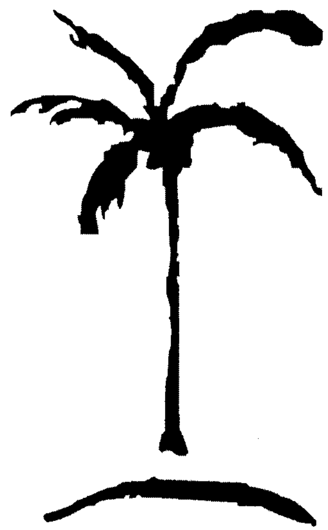


Mathematics

Mathematics

Year 13

Rory Barrett



Government of Samoa
MINISTRY OF EDUCATION, SPORTS AND CULTURE

Mathematics
Year 13
Samoa Secondary School Curriculum

ISBN: 0-9583564-6-7

Acknowledgements

This textbook is based on *Year 12 Mathematics Study Guide*, by Rory Barrett, and ESA Publications (NZ) Ltd, Auckland. The Mathematics Subject Committee wishes to thank Mr Mark Sayes, Director, ESA Publications (NZ) Ltd for his permission to republish this title and for his assistance throughout the production process.

Mathematics Subject Committee

La Lea'ana
Aperila Va'alotu
Solomona Ah Lam
Penitala Kolose
Mike Webster
Naomi Fili
Tasi Falefatu

Talalelei Soloi was the CMAD subject adviser and Dr Brian Bennison was the New Zealand consultant.

Produced as part of the Samoa Secondary Education Curriculum and Resources Project for the Ministry of Education, Sports and Culture, Samoa, 2004.

Funded by the New Zealand Agency for International Aid (NZAID), Nga Hoe Tuputupu-mai-tawhiti.

Printed in Singapore

Managing Contractor: Auckland UniServices Ltd.

To the Teacher

This book contains the entire year 13 requirements for Mathematics in Samoa Secondary Schools.

The level number and objectives at the front of each chapter refer to the New Zealand Curriculum for which this title was originally published. Teachers must refer to the Mathematics Curriculum Statement for Samoa Secondary Schools during their planning of units of work and ensure that students are fully aware of the achievement objectives from this document.

CONTENTS

1. BASIC ALGEBRA

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 5 ALGEBRA

- *to evaluate linear expressions by substitution*
- *to combine like terms in algebraic expressions*

LEVEL 6 ALGEBRA

- *to substitute values into formulae*

LEVEL 7 ALGEBRA

- *to carry out appropriate manipulation and simplification of algebraic expressions*

Introduction and Conventions

Basic algebra needs to be understood and practised so that problems can be accurately and quickly solved.

If x and y represent two numbers then:

a. $x + y$ represents the number obtained by **adding** y to x.
b. $x - y$ represents the number obtained by **subtracting** y from x.
c. xy represents the number obtained by **multiplying** x and y.
d. $\frac{x}{y}$ represents the number obtained by **dividing** x by y.
e. x^2 means $x \times x$; in the term x^2, 2 is the **power** (**or exponent or index**) of the base x.

Note: xy can also be represented by $x \times y$.
$\frac{x}{y}$ can also be represented by $x \div y$.

Using the above, $5x^2y + 2x^3$ represents $5 \times x \times x \times y + 2 \times x \times x \times x$.

Substitution

A **formula** is an equation which uses variables and symbols to describe a mathematical relationship. **Substitution** means replacing the variables with numbers, and usually results in a calculation.

Example A: Find the value of C in the formula $C = 2A - 4B$ when $A = 5$ and $B = 2$.

Solution: substituting $A = 5$ and $B = 2$

gives $C = 2 \times 5 - 4 \times 2$

$= 10 - 8$

$\therefore C = 2$

Example B: The formula for the area of a circle is $A = \pi R^2$ where A is the area and R is the radius. Find the area to 2 **decimal places**, if $\pi = 3.142$ and $R = 8.215$cm.

Solution: $A = \pi R^2$

$\therefore A = 3.142 \times (8.215)^2 \text{ cm}^2$ [substituting]

$= 212.04\text{cm}^2$ (2 decimal places)

Example C: $V = 3x^2 - 8y^2$. Find V when $x = -3$, $y = \frac{1}{2}$.

Solution: $V = 3(-3)^2 - 8(\frac{1}{2})^2$

$\therefore V = 27 - 2$

$\therefore V = 25$

Note: In problems involving the substitution of negative numbers and fractions it is essential to use brackets when making the substitutions. Thus in Example C:

$V = 3 \times -3^2 - 8 \times \frac{1}{2}^2$ [removing brackets too soon]

$= -27 - 4$

$= -31$, which is wrong [if calculated as written down]

Exercise 1a

1. Evaluate the following expressions:
   - a. $A \times B$ if $A = 3, B = 7$
   - b. BC if $B = 4, C = 5$
   - c. 2PQ if $P = 4, Q = 6$
   - d. 3PQ if $P = 5, Q = 7$
   - e. $7A \times 5B$ if $A = 10, B = 9$
   - f. 4P if $P = -3$
   - g. 5QR if $Q = -3, R = 4$
   - h. 7PA if $P = -2, A = -3$
   - i. ABC if $A = 3, B = 4, C = -1$
   - j. $2A \times 3B$ if $A = -3, B = -4$

2. Evaluate:
   - a. $2P + 3Q$ if $P = 4, Q = -2$
   - b. $PQ + 5$ if $P = -5, Q = 3$
   - c. $3P + 7$ if $P = -4$
   - d. $A - B$ if $A = 7, B = 9$
   - e. $A - 2B$ if $A = -3, B = 8$
   - f. $3P - Q$ if $P = -2, Q = 5$
   - g. $8Q - P$ if $Q = 3, P = -5$
   - h. $6 - (P + Q)$ if $P = 3, Q = 9$
   - i. $2P - B + 3$ if $P = 4, B = -5$
   - j. $AB + C$ if $A = -3, B = -4, C = 5$

3. Evaluate the following:
   - a. A^2 if $A = 3$
   - b. $4X^2$ if $X = 2$
   - c. $4P^3$ if $P = 2$
   - d. $4P^2Q$ if $P = 3, Q = 2$
   - e. $A^3 + A^2$ if $A = 2$
   - f. $A^3 + B^2$ if $A = 1, B = 5$
   - g. $X^3 - X^2$ if $X = 5$
   - h. $3(A^2 + B^2)$ if $A = 1, B = 2$
   - i. $(P - Q)^3 - 1$ if $P = 4, Q = 1$
   - j. A^2B^3 if $A = 4, B = 2$
   - k. X^2 if $X = -4$
   - l. $2A^2$ if $A = -4$
   - m. $3X^2 - 2Y^2$ if $X = 5, Y = 10$
   - n. $5X + X^2$ if $X = -3$
   - o. $(A + B)^2$ if $A = 4, B = -7$

4. Evaluate the following:
   - a. $\frac{A+B}{3}$ if $A = 2, B = 4$
   - b. $6AP \div 3$ if $A = 4, P = 7$
   - c. $\frac{(A-B)^2}{3}$ if $A = 4, B = 1$
   - d. $\frac{9QP}{Q}$ if $P = 5, Q = 2$
   - e. $(2A + 4B) \div 7$ if $A = 14, B = 7$
   - f. $\frac{AB}{2C}$ if $A = 4, B = 9, C = 3$
   - g. $B^2 \div C$ if $B = 4, C = 2$
   - h. $\frac{3y^2}{y^3}$ if $y = 3$
   - i. $\frac{2A+B}{C}$ if $A = 4, B = 6, C = 7$
   - j. $\frac{B \times C}{B+D}$ if $B = 3, C = 9, D = 1$
   - k. $\frac{A-9}{2}$ if $A = 3$
   - l. $\frac{BC}{A}$ if $B = -2, C = -9, A = 3$
   - m. $\frac{A^2}{C}$ if $A = 9, C = -3$
   - n. $\left(\frac{P}{Q}\right)^2$ if $P = -8, Q = -2$

Repeated Substitution by Calculators

Many substitution problems can be done more easily using an electronic calculator. Often it is desirable to do the same calculation with a variety of different values.

Although this can be done using any scientific calculator, the task is enormously simplified if the calculator is programmable.

Example D: Calculate the value of $2A + 3B$ over a large number of values. The following program in the programming mode of any Casio programmable calculator will enable this repeated calculation to be done very simply.

```
? → A : ? → B : 2 × A + 3 × B
```

On pressing the [**EXE**] key the symbol ? appears, inviting the value of A to be typed in.

The **[EXE]** button is pressed and another ? appears. The value of B is entered and **[EXE]** is pressed. This causes the answer to be displayed.

Pressing [EXE] again restarts the program for the next values of A and B.

Students using programmable calculators made by other companies will have to refer to the owner's manual to get the equivalent program.

Exercise 1b

1 a. What changes would be necessary in the program in Example D to get a program which would do repeated calculations of 4A – B + 2C?
b. How many times would ? appear when the program was run once?

2. a. What changes would be necessary in the program in Example D to get a program which would do a repeated calculation of $3A^4$?
b. How many times would ? appear when the program was run once?

Simplifying

Simplifying means writing an expression as simply as possible. It is important to simplify in order to make the processes of substitution and calculation swift.

An **algebraic term** is made up of one or more **variables**, a **coefficient** and one or more **exponents**. Variables are letters which represent numbers. (The value of variables can change.) A number multiplying the variables is called the *coefficient of the variable*. An exponent is also called a **power** or **index**.

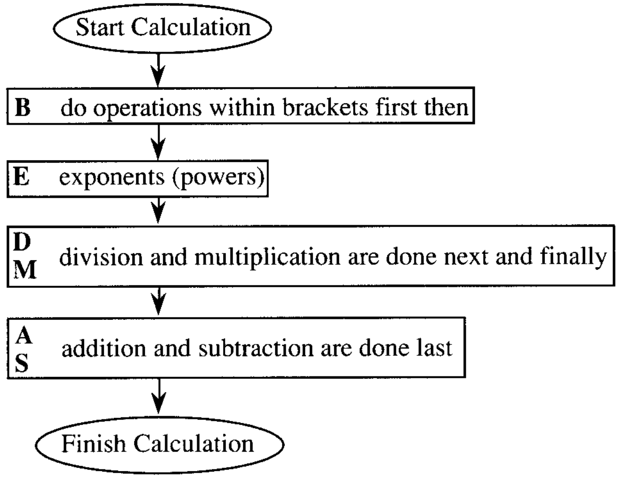
Example E:
For the expression $3x$, the variable is x and the coefficient of x is 3.
For the expression $8a^2$, the variable is a, coefficient is 8, and power of a is 2.
For the expression $2a^3b^8$, the variables are a and b, the coefficient is 2, the power of a is 3 and the power of b is 8.

Like terms are terms with the *same* variables and the same powers. Thus $2x$ and $14x$, $3y^2$ and $4y^2$ are **like terms**, but $2x^2$ and $3x^3$, $4xy$ and $16x^2y$ are **unlike terms**. Unlike terms cannot be combined into a simpler expression when they are added or subtracted together.

If a, b and c are any *real numbers* then the following laws are true:

	Addition	Multiplication
Commutative Law	a + b = b + a	a × b = b × a
Associative Law	a + (b + c) = (a + b) + c	a × (b × c) = (a × b) × c
Distributive Law	a × (b + c) = a × b + a × c	

The order of arithmetical operations is given by the mnemonic **BEDMAS** and is explained below.



Note:
a. If several of the operations, addition and subtraction are involved they are done in order from left to right. The same is true for multiplication and division.
b. If more than one set of brackets is involved, they are removed in order from left to right.

Example F:

a. 2a + 5a = 2 × a + 5 × a
= (2 + 5) × a [distributive law]
= 7 × a [adding coefficients]
= 7a

b. 11a – 3a = 11a + $^{-}$3a [subtracting is 'adding the opposite']
= 8a [adding coefficients]

Note: In practice, the coefficients of a are added together.

c. $2a \times 3b = 2 \times a \times 3 \times b$
$= 2 \times 3 \times a \times b$ [commutative law]
$= (2 \times 3) \times (a \times b)$ [associative law]
$= 6 \times a \times b$
$= 6ab$

Note: In practice, the coefficients are multiplied together.

d. $3a \times 4b + 2 \times ab + 4ad = 12ab + 2ab + 4ad$ [simplifying]
$= 14ab + 4ad$ [adding 12ab and 2ab]

Note: Further simplification is not possible because 14ab and 4ad are unlike terms and cannot be added together to give a simpler expression.

e. $12a \div 3 = 12a \times \frac{1}{3}$ [division is replaced by multiplication by the reciprocal]
$= 12 \times \frac{1}{3} \times a$
$= 4 \times a$ [since $12 \times \frac{1}{3} = 4$]
$= 4a$

Note: In practice, all that is required is to divide 12 by 3.

f. $AB \div B = AB \times \frac{1}{B}$ [division is the same as multiplication by the reciprocal]
$= A \times B \times \frac{1}{B}$
$= A \times 1$ ['cancelling' B]
$= A$

Note: In practice, all that is required is to cancel out the B's: $AB \div B = \frac{AB}{B} = A$.

Exercise 1c: Basic Problems

1. Simplify the following expressions as much as possible:

a.	$3x + 4x$	b.	$5A - 3A + 2B + 3B$
c.	$11Q + 4Q + 5P + 6Q$	d.	$(6P + 3Q + 2Q) + (4P + 2Q)$
e.	$11P - 5P$	f.	$2x + 3x + 2y$
g.	$6A + 9B - 2A$	h.	$12P - (2P + 7P)$
i.	$(9A - 2A) + 3B + 2A$	j.	$18Q + 24B - 7B - 9P$

2.

a.	$4 \times B$	b.	$A \times E$	c.	$B \times 5 \times 4$
d.	$3 \times A \times 4 \times B$	e.	$4 \times A \times 7$	f.	$12 \times C \times D \times 3$
g.	$4 \times 5 \times Q \times A$	h.	$8B \times 6A$	i.	$3A \times 5BP$
j.	$12C \times 5PQ \times 4R$				

3.

a.	$6A \div 2$	b.	$12B \div 3$	c.	$5A \div A$
d.	$15AB \div 5A$	e.	$36ABC \div 6AC$	f.	$\frac{8B}{4}$

g.	$\frac{ABC}{AB}$	h.	$\frac{AC}{AB}$	i.	$\frac{5B}{10}$
j.	$\frac{A}{AC}$	k.	$\frac{AB}{3BC}$	l.	$\frac{3}{A}$
m.	$\frac{6}{2B}$				

Powers or Exponents

Products of the same number or variable can be simplified by writing them in **power form** as illustrated in the following example. Powers are also called exponents or indices.

Example G: $2 \times 2 \times 2 \times 2 = 2^4$ [2 is the base, 4 is the power]
$A \times A \times A = A^3$ [A is the base, 3 is the power]
$2A \times 2A \times 2A = 2^3 \times A^3 = 2^3A^3$

Products of powers of the *same* base can be written as the base to the *sum* of the powers:

$$A^mA^n = A^{m+n}$$

Example H: a. $A^3A^5 = A^{3+5}$
$= A^8$

b. $8 \times 16 = 2^3 \times 2^4$
$= 2^7$

$$(A^m)^n = A^{mn}$$

Example I: a. $(P^3)^5 = P^{3\times5}$
$= P^{15}$

b. $(2^3)^4 = 2^{3\times4}$
$= 2^{12}$

$$(AB)^m = A^mB^m$$

Example J: $(3x)^3 = 3^3x^3$
$= 27x^3$

Example K: Simplify: $A^3A^7 + 3A^2 \times 5A^8 + A^3 \times A^{11} + 6A^{10}$

Solution: $A^3A^7 + 3A^2 \times 5A^8 + A^3 \times A^{11} + 6A^{10}$
$= A^{10} + 15A^{10} + A^{14} + 6A^{10}$ [evaluating products]
$= 22A^{10} + A^{14}$ [adding all the A^{10} terms]

Note: The expression $22A^{10} + A^{14}$ cannot be simplified any further because $22A^{10}$ and A^{14} are unlike terms and cannot be added together to give a simpler expression.

$$A^m \div A^n = \frac{A^m}{A^n} = A^{m-n}$$

Example L: $A^{11} \div A^3 = A^{11-3}$
$= A^8$

Example M: $\frac{36A^{11}}{24A^8} = \frac{3A^{11-8}}{2}$ [dividing numerator and denominator by 12 and the smallest power of A which is A^8]

$= \frac{3A^3}{2}$

Example N: Simplify $\frac{25x^{15}y^4}{30x^{10}y^7}$

Solution: The numerator and denominator have common factors 5, x^{10} and y^4.

$\frac{25x^{15}y^4}{30x^{10}y^7} = \frac{25x^{10}.x^5.y^4}{30x^{10}.y^3.y^4}$ [as $x^{15} = x^{10}.x^5$] [as $y^7 = y^3.y^4$]

$= \frac{25x^5}{30y^3}$ [cancelling x^{10}, y^3]

$= \frac{5x^5}{6y^3}$ [simplifying]

$x^0 = 1$ and $\frac{1}{x^n} = x^{-n}$ so long as $x \neq 0$

Example O: If $x \neq 0$ show that: a. $x^0 = 1$ and
b. $\frac{1}{x^n} = x^{-n}$ (if $x \neq 0$)

Solution: a. $x^0 = x^{n-n}$
$= x^n \div x^n$ [dividing by x^n since $x \neq 0$]
$= 1$

b. $\frac{1}{x^n} = \frac{x^0}{x^n}$ [since $x^0 = 1$]
$= x^{0-n}$ [subtracting indices]
$= x^{-n}$

Exercise 1d

1. Simplify as much as possible:
   a. $A \times A$ b. $A \times A \times A \times B \times B$ c. $P \times Q \times P \times Q$
   d. $2 \times A \times A$ e. $P \times 3 \times 4 \times P \times Q$ f. $6P \times 7P \times Q$
   g. $6P \times 5P \times 6Q \times Q$ h $5A \times 9 \times 6$ i. $3A \times A \times B$
   j. $4L \times 3L \times L$ k. $A \times A^2$ l. $2 \times A \times 3 \times 4 \times A^2$
   m. $P^2 \times Q \times Q \times P^2$ n. $3M^2 \times 4M^2$ o. $ABC \times BC$

2. Simplify:
   a. $5A + 2B + 3A$ b. $7P - 10P$
   c. $5AB + 3AB + BC + 2BC$ d. $3A + 2A + 3A + 2AB$
   e. $4xy - 7xy$ f. $3A + 2B - 8A - 4B$
   g. $4A^2 + 5A + 2A + 3A^2$ h. $AB + BC + 3AB - BC$
   i. $6PQ - 3PQ + P - 4P$ j. $15 - 3x + 2 - 3x$
   k. $7P - 13 + 3P + 2$ l. $4B + 3A + 2C - B + 8A$
   m. $A - B + 3B - 3$ n. $4PQ + AB + 6AB - 11PQ$
   o. $6A^2 - 3 + 4A^2 - 4$

3. Simplify the following:
   a. $5A \div 5$ b. $\frac{9x}{3}$ c. $5AB \div B$
   d. $49PA \div 7Q$ e. $\frac{10A^2C}{C}$ f. $\frac{ABC}{BC}$
   g. $A^2 \div A$ h. $5M^2 \div 5M$ i. $1000M^2N \div 25M$
   j. $\frac{10P^2}{5P}$ k $64P^2Q^2 \div PQ^2$ l. $50M^2 \div 10$
   m. $\frac{8AB^2}{4AB}$

4. Simplify:
   a. $5A \times 3B$ b. $2A \times (3A \times A)$ c. $4A \times 6B \times P$
   d. $A^2 \times A^3$ e. $4P^3 \times 5P^{11}$ f. $(6P \times 3P) \times 2Q$
   g. $(3P)^2 \times P$ h. $(4R)^2 \times 3R$ i. $(3P)^2 \times (2A)^3$
   j. $ABC \times 2ABD$ k. $5A \div A$ l. $8AB \div 2A$
   m. $12x^2 \div 3x$ n. $48x^2 \div 12xy$ o. $25x^5 \div 25x^2$
   p. $\frac{A^2}{A}$ q. $\frac{4A^2B}{2AB}$ r. $\frac{36AB^3}{3AB}$
   s. $\frac{2}{4A}$ t. $\frac{4B}{8AB}$

Exercise 1e: Further Simplification

Simplify the following (where possible):

1. $6x + 12x$ 2. $6x \times 12x$ 3. $3x^6 + 4x$
4. $2x^6 \times 5x^2$ 5. $6x^6 \times 2x^2$ 6. $(x^6)^2 \times 3x^{11}$
7. $6y + 13y$ 8. $6yx^2 \times 3yx^3$ 9. $2y^6 + 3y$
10. $5y^6 \times 6y^3$ 11. $6y^6 \times 8y^3$ 12. $(y^6)^3 \div 2y^{15}$

13. $18a + 2a$	14. $8a^2 \times 2a^3$	15. $3a^8 + 2a^2$
16. $a^8 \times 3a^2$	17. $8a^8 \times 3a^2$	18. $(a^8)^2 \times 3a^2$
19. $18b + {}^-4b$	20. $3b \times (4b)^2$	21. $5b^8 \div b^4$
22. $(2b)^8 \times 2b^4$	23. $18b^8 \times 4b^4$	24. $(b^8)^4 \div b^{13}$
25. $(3x^3)^2 \div 3x^2$	26. $9x^8 \div 3x^5$	27. $13x \div 12y$
28. $4x^2 + 5xy$	29. $6x^2 - 13x$	30. $4xy - 2yx$
31. $2x^2 + 5x^2$	32. $-2x + 15x$	33. $-2x^2 \div 5x^3$
34. $(2x)^2 \times 15x$	35. $-2x \times 5x$	36. $(3y)^2 + (4y)^2$
37. $(3y)^2 - 4y^2$	38. $-3y + (4y)^2$	39. $-3y \div 4y$
40. $(3y^2)^2 \times 4y$	41. $3y \times (-4y)^2$	42. $-3y^2 \times 4y$
43. $(-3y)^2 \times -4y^3$	44. $2x^6 + 3x^2 - x^4$	45. $(6x)^2 + (2x)^2$
46. $\dfrac{(2x)^6 \times (2x)^2}{(4x)^4}$	47. $\dfrac{(6x)^2 \times (2x)^3}{(4x)^2}$	48. $\dfrac{(3x)^2 \times (2x^2)^2}{4x^2}$
49. $\dfrac{(3x^2)^2 \times (2x^3)^2}{9x^5}$	50. $\left(\dfrac{(5x)^2 \times 3x}{15x^2}\right)^2$	

Problems and Investigations

1. Use 4 x's and the numbers 2, 3, 5, 7 to obtain calculations which give the following:

   $13x$, x^{17}, $x^3 + 2x^2$, $x^8 + x^5$, x^2, $2x^{10}$, 0, $2x^{12} + 3x^2$, $41x^2$, $30x^3 + 7x$

   [Eg: Find a calculation which gives $105x^3 + 2x$. Answer: $3x \,.\, 5x \,.\, 7x + 2x$]

2. $$P = \frac{x^2 + y^2 + z^2}{x^3 + y^3 + z^3 + xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2}$$

   where x, y, z can take values from 1, 2, 3, ... 9. It is claimed that P is always the reciprocal of a natural number. Investigate this.

3. a. Investigate the expressions $2^A 12^B$ and $4^C 3^D$ and determine if they can ever have the same value.

   b. If it is possible for equality to occur, find the relationships between A, B, C, D.

4. a. Investigate the following expressions by substituting a variety of values for m and n and finding a simpler form in which they can be expressed.

   i. $\dfrac{(32x^2)^n}{16^n.(2x)^{n-1}}$ ii. $\dfrac{6^{n-1}x^{m+1}}{3^n 2^{n-2} x^{m-1}}$ iii. $\dfrac{10^{n-2}x^{m+3}}{5^{n-1}2^{n-3}x^{m-2}}$

   iv. $\dfrac{(12x)^{n-1}}{2^{2n}3^{n-3}x^{n+1}}$ v. $\dfrac{16^{m-1}(3x)^m}{2^{4m}9^{m-1}x^{m-3}}$

   b. Prove your simplification.

2. LINEAR EQUATIONS

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 5 ALGEBRA

- *to solve linear equations*

LEVEL 6 ALGEBRA

- *to form and solve linear equations*

LEVEL 7 ALGEBRA

- *to write appropriate equation(s) or inequation(s) to describe a practical situation*

Introduction

An **equation** is a mathematical sentence which is true for certain numbers.
In a **linear** equation, the power of the variable is 1.

Example A: $x + 3 = 5$ is a linear equation which is true for $x = 2$ but false for any other values of x such as $x = 4$ or $x = -1$.

Solving Equations

Solving an equation means finding the numbers which make the equation true. The basic method for solving linear equations is to *isolate the variable on one side of the equals sign.* This method involves "undoing" operations by doing the inverse operation to *both sides* of the equation.

Example B: The variable x in the equation $x + 5 = 2$ is isolated by removing the 5 from the left-hand side to isolate the x. This is done by *subtracting* 5 from *both sides* of the equals sign.

$x + 5 = 2$

$\therefore\ x + 5 - 5 = 2 - 5$ [subtracting 5 from both sides]

$\therefore\ x = -3$ [simplifying]

Note: Subtracting 5 from *both* sides keeps both sides of the equation equal to each other.

Example C: S

Example C: Solve the following equations.

a. $2A = 7$

$\frac{2A}{2} = \frac{7}{2}$ [dividing both sides by 2]

$\therefore A = 3\frac{1}{2}$ [simplifying]

b. $5R - 3 = 9$

$5R = 12$ [adding 3 to both sides]

$\therefore R = \frac{12}{5}$ [dividing both sides by 5]

$\therefore R = 2.4$

c. $\frac{3A + 5}{2} = 7A$

$\therefore 3A + 5 = 14A$ [multiplying both sides by 2]

$\therefore 11A = 5$ [subtracting 3A from both sides and swapping sides]

$\therefore A = \frac{5}{11}$ [dividing both sides by 11]

d. $8 - 3x = 7$

$\therefore 8 + -3x = 7$ [changing $-3x$ to $+ -3x$]

$\therefore -3x = -1$ [subtracting 8 from both sides]

$\therefore x = \frac{-1}{-3}$ [dividing both sides by -3]

$= \frac{1}{3}$

Exercise 2a

1. Solve each of these equations:
   a. $3x = 7$ b. $11P = 23$ c. $8C = 3$
   d. $Z + 3 = 2$ e. $Z + 2.7 = 8$ f. $\frac{x}{3} = 1.2$
   g. $\frac{x}{1.3} = -2.1$ h. $-3P = -2$ i. $P - 12 = -4$
   j. $2x = 1.1$

2. Solve each of these equations:
   a. $4x - 1 = 7$ b. $2P + 11 = -2$ c. $-2A + 3 = 14$ d. $-2P + 11 = -3$
   e. $-8 - 3A = 4$ f. $\frac{x}{3} + 5 = 7$ g. $\frac{P}{3} - 7 = 2$ h. $2 - \frac{P}{3} = 4$
   i. $8 - 3A = \frac{1}{2}$ j. $19 - 3A = \frac{1}{2}$

3. Solve each of these equations:
   a. $\frac{2}{3}x = 8$ b. $\frac{4}{5}P = -1$ c. $\frac{7R}{2} = -5$ d. $\frac{-5R}{4} = -9$
   e. $\frac{4P}{5} + 11 = 2$ f. $\frac{3R}{4} = \frac{4}{7}$ g. $\frac{4P}{7} - 3 = 2$ h. $\frac{5}{2}B - 3 = -1$
   i. $\frac{3}{4}x + 12 = 2.4$ j. $P + \frac{4}{3} = 5$

4. Solve each of these equations:
   a. $x + x + 2x = 7$ b. $3x - 4 = 0$ c. $5A - 2A + 5 = 12$
   d. $4(P - 3) = 19$ e. $6(R - 3) = 19$ f. $5(R + 3) + 2(R - 3) = 25$
   g. $5P + 2(P + 1) = 12$ h. $8P - (3 + P) = 12$
   i. $17 - 2(P + 3) = 25$ j. $28 - (1 - P) = 42$

5. Solve each of these equations:
   a. $6x = 3x + 12$ b. $4x = 2x - 13$
   c. $5A = 17 + 3A$ d. $3P + 1 = 2P + 11$
   e. $15R - 11 = 5R + 13$ f. $16R + 11 = 2R + 47$
   g. $6R + 12 = 2R - 19$ h. $6(R - 3) = 3R + 25$
   i. $5(x + 7) = 3(x - 2)$ j. $8(P + 3) - 3P = 4(P + 1) - 12$

Note: In all remaining examples reference to 'both sides' will be omitted.

Example D: Solve the following equations.

i. $\frac{5}{3x + 1} = 2$

$\therefore 5 = 2(3x + 1)$ [multiplying by $(3x + 1)$]

$\therefore 5 = 6x + 2$ [expanding]

$\therefore 3 = 6x$ [subtracting 2]

$\therefore 6x = 3$

$x = \frac{3}{6}$ [dividing by 2]

$= \frac{1}{2}$

ii. $2\frac{1}{2}(x - 1) - \frac{x + 3}{3} = 4$

$5(x - 1) - \frac{2x + 6}{3} = 8$ [multiplying by 2]

$\therefore 15(x - 1) - (2x + 6) = 24$ [multiplying by 3]

$\therefore 15x - 15 - 2x - 6 = 24$ [expanding]

$\therefore 13x - 21 = 24$ [simplifying]

$\therefore 13x = 45$ [adding 21]

$x = \frac{45}{13}$ [dividing by 13]

$= 3\frac{6}{13}$

Exercise 2b

1. Solve:
   a. $5z + 3z + 11 = 12$
   b. $3(L + 2) + 5L = 27$
   c. $7(2x + 1) + 5(x - 3) = 12$
   d. $9(T + 3) - 5(T + 4) = 8$
   e. $7 - 2(3 - 2R) = 38$
   f. $5(2T + 3) = 2T + 4$
   g. $4(2L + 1) = 3(4 - L)$
   h. $\frac{1}{2}(3R + 1) = 4$
   i. $5(3R + 1) + 3(2R + 7) = 11R + 42$
   j. $5T + 3(T + 2) = 2(T + 1)$
   k. $4P - 3(P - 4) = 5(P + 11) + 8$
   l. $4(P + 1) + 2(2P + 3) = 4(P + 2) - 3(P - 1)$

2. Solve:
   a. $\frac{x}{3} + \frac{x}{4} = 2$
   b. $\frac{2T}{3} - \frac{T}{4} = \frac{1}{6}$
   c. $\frac{3R + 1}{4} + \frac{2R - 3}{2} = 4$
   d. $\frac{3R + 1}{2} = 2R$
   e. $\frac{P + 1}{3} + \frac{P - 3}{2} = \frac{P + 4}{6}$
   f. $\frac{2P + 1}{4} - \frac{P - 1}{3} = \frac{1}{6}$
   g. $\frac{3}{4}(A + 2) = \frac{2}{3}(A - 3)$
   h. $\frac{2(A + 1)}{3} = \frac{A - 1}{5}$
   i. $\frac{3}{5}(A + 1) + \frac{2}{3}(A - 1) = 2$
   j. $\frac{A + 1}{A} = 2$
   k. $\frac{3A - 1}{2A + 5} = \frac{1}{2}$
   l. $\frac{A + 3}{A} = \frac{2A - 1}{A}$
   m. $3 - 2(A - 3) = 4A - 3$

Solving Equations using Calculators with Equation-solving Modes

Certain calculators enable the student to enter an equation directly and press a button to get the answer. One of the most powerful is the Hewlett Packard 48G.

Example E:

Solve $\frac{2x + 3}{4} - \frac{3x - 7}{2} + \frac{4x + 5}{6} = 2$ to 3 decimal places.

Solution: These instructions would apply for an HP 48G. Students using other types of calculator would need to refer to their owner's manual.

First step: press 'solve'
Second step: enter 'solve equation'
Third step: type in the equation
Note: $\frac{2x + 3}{4}$ is entered as $(2 \times x + 3) \div 4$
Fourth step: select 'solve'
Answer: 9.25

Exercise 2c

If the student has a calculator with an equation-solving facility, he or she should practise solving some of the equations in Exercise 2b. For simpler examples, it is quicker to work equations out by hand but for complicated examples it is definitely quicker and usually more accurate to solve them using a calculator with this facility.

Solving Equations using Spreadsheets

Equations of any type may be solved to whatever degree of accuracy is required by the use of **spreadsheets**.

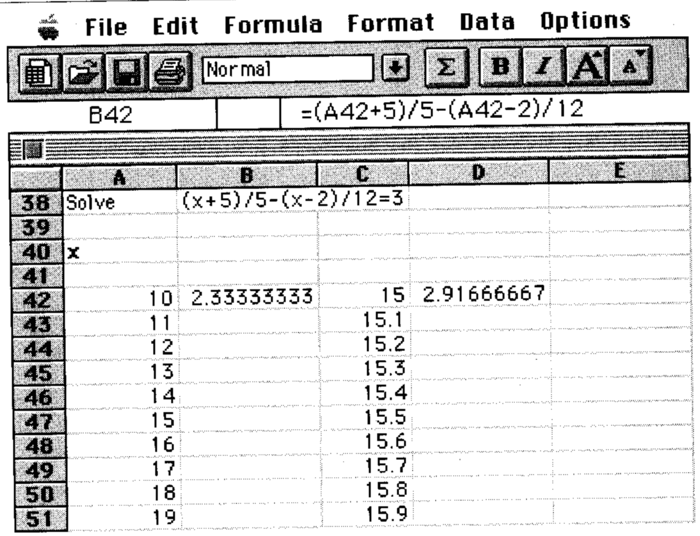
Example F:

Solve $\frac{x + 5}{5} - \frac{x - 2}{12} = 3$

Solution: By trial and error a value for x is found which, when substituted into the formula $\frac{x + 5}{5} - \frac{x - 2}{12}$, gives a result which is quite close to 3.

$\frac{(10 + 5)}{5} - \frac{(10 - 2)}{12} = 2\frac{1}{3}$ [substituting $x = 10$]

This value of x gives a starting point for a series of x values which are found in the cells A42 – A51 in the diagram below, taken from Microsoft Excel for Macintosh.



	A	B	C	D	E
38	Solve	(x+5)/5-(x-2)/12=3			
39					
40	x				
41					
42	10	2.33333333	15	2.91666667	
43	11		15.1		
44	12		15.2		
45	13		15.3		
46	14		15.4		
47	15		15.5		
48	16		15.6		
49	17		15.7		
50	18		15.8		
51	19		15.9		

	A	B	C	D	E	F
38	Solve	(x+5)/5–(x–2)/12=3				
39						
40	x					
41						
42	10	2.33333333	15	2.91666667	15.7	2.99833333
43	11	2.45	15.1	2.92833333	15.71	2.9995
44	12	2.56666667	15.2	2.94	15.72	3.00066667
45	13	2.68333333	15.3	2.95166667	15.73	3.00183333
46	14	2.8	15.4	2.96333333	15.74	3.003
47	15	2.91666667	15.5	2.975	15.75	3.00416667
48	16	3.03333333	15.6	2.98666667	15.76	3.00533333
49	17	3.15	15.7	2.99833333	15.77	3.0065
50	18	3.26666667	15.8	3.01	15.78	3.00766667
51	19	3.38333333	15.9	3.02166667	15.79	3.00883333

The cell B42 is selected and the formula = (A42 + 5)/5 – (A42 – 2)/12 is typed into the formula bar. This formula is then entered into the cells B42 to B51 giving the values of $\frac{(x+5)}{5}-\frac{(x-2)}{12}$ when the values of x listed in the A column are substituted into that expression.

It can be seen that in cell B47 the value is less than 3 and in B48 it is more than 3; hence, we deduce the root lies between 15 and 16.

A further series of x values are inserted in cells C42 to C51 going from 15 to 15.9. The corresponding $\frac{(x+5)}{5}-\frac{(x-2)}{12}$ values are in the cells to the immediate right. Inspecting these shows the root to be between 15.7 and 15.8. This process is repeated once more showing the root to lie between 15.71 and 15.72; hence a 1 decimal place approximation is 15.7.

Exercise 2d

	A	B	C	D	E	F
64	Solve	(2x+4)/5–(x–2)/8=3				
65						
66	x					
67						
68	1	1.325	7	2.975	7	2.975
69	2	1.6	7.1	3.0025	7.01	2.97775
70	3	1.875	7.2	3.03	7.02	2.9805
71	4	2.15	7.4	3.0575	7.03	2.98325
72	5	2.425	7.5	3.085	7.04	2.986
73	6	2.7	7.6	3.1125	7.05	2.98875
74	7	2.975	7.7	3.14	7.06	2.9915
75	8	3.25	7.8	3.1675	7.07	2.99425
76	9	3.525	7.9	3.195	7.08	2.997
77	10	3.8	8	3.2225	7.09	2.99975
78	11	4.075		3.25	7.1	3.0025

1. The above spreadsheet shows a partial solution to an equation. What is the equation which is being solved?
2. To how many decimal places can we give the solution to the equation?
3. Between what two integers does the root lie?
4. What was typed into the formula bar?
5. What value would be in cell B71 if, instead of solving the above equation, we solved $\frac{(2x+5)}{6}-\frac{(x-2)}{9}=3$ using the same values in cells A68 to A78?

Solving Equations using Programmable Calculators

Example G: Solve $\frac{x+5}{5}-\frac{x-2}{12}=3$ using a programmable calculator without an equation solving mode.

Solution: By trial and error the root will be found to lie between 15 and 16. The reason for this is that when 15 is substituted a value less than 3 is obtained and when 16 is substituted a value greater than 3 is obtained. The root must lie between two of the numbers 15.0, 15.1, 15.2, . . . , 15.9, 16.0. Trial and error shows it to be between 15.7 and 15.8.

The root must lie between two of the numbers 15.70, 15.71, 15.72, . . . , 15.79, 15.80. Trial and error shows it to lie between 15.71 and 15.72.

The root must lie between 15.710, 15.711, 15.712, . . . , 15.719, 15.720. Trial and error shows it to lie between the numbers 15.714 and 15.715.

Continuing this process will allow the user to get whatever degree of accuracy is required.

Exercise 2e

1. What changes would be made to the program if the solution to $\frac{x+7}{2} - \frac{x-3}{4} = 8$ was required?

2. What changes would be made to the first program if a solution was required correct to 5 decimal places?

3. Suppose an equation was being solved, and the solution was $6\frac{2}{13}$. What would be the solution using the program in the example, if the only change was in the formula for the equation?

Word Problems

Problems given in words (**word problems**) need to be changed to a mathematical equation to be solved. This process is called **mathematical modelling**. The general approach is illustrated by the following example.

Example H: Shane is taller than Jason by 2.4cm. Jason is taller than Ian by 1.3cm. Their combined height is 452cm. What are their heights?

Solution: Let Ian's height be h
∴ Jason's height is h + 1.3 [Jason is 1.3cm taller than Ian]
Shane's height is h + 1.3 + 2.4 = h + 3.7 [Shane is 2.4cm taller than Jason]
Since their combined height is 452cm we have:

$$h + h + 1.3 + h + 3.7 = 452$$
$$\therefore\ 3h + 5 = 452 \quad \text{[simplifying]}$$
$$\therefore\ h = 149 \quad \text{[subtracting 5 and dividing by 3]}$$

Thus Ian is 149cm tall; Jason's height is 149 + 1.3 = 150.3cm and Shane's height is 149 + 1.3 + 2.4 = 152.7cm.

Example I:

Solution of Example H using a spreadsheet:

	A	B	C	D
1	Three Boys Problem			
2				
3	Jason	Shane	Ian	Total
4	101.3	103.7	100	305
5	151.3	153.7	150	455
6	149.3	151.7	148	449
7	150.3	152.7	149	452

This solution was found using the spreadsheet Microsoft Excel, as were the others in this chapter. Similar results will be obtained by using identical methods on most commonly-used spreadsheets.

The cell, A4, was selected and into the formula bar was typed = C4 + 1.3; then this was entered into the column below Jason (this formula relates Ian's height to Jason's). Similarly, into B4 and the cells below it was entered =A4 + 2.4 (relating Jason's height to Shane's). By a quick process of trial and error (entering different values for Ian) the correct solution is found when Ian's height is 149cm .

	A	B	C	D
1	Three Boys Problem			
2				
3	Jason	Shane	Ian	Total
4	101.3	103.7	100	305
5				
6				
7				

	A	B	C	D
1	Three Boys Problem			
2				
3	Jason	Shane	Ian	Total
4	101.3	103.7	100	305
5	151.3	153.7	150	455
6				
7				

Example J:

Solution of Example H using a programmable calculator.

With a Casio calculator the following program will solve the problem (owners of other brands check their owner's manual).

$$? \rightarrow : I + 1.3 \rightarrow J : J + 2.4 \rightarrow S : I + S + J \rightarrow T$$

By keeping a record of the values input for Ian's height, the correct value can quickly be found.

Exercise 2f: Solving Word Problems

For each of the following, find: (i) an equation; (ii) a solution using your equation or any other method.

1. Item B costs $1 more than Item A. Sandra purchases 5 of type A and 6 of type B. Her total expenditure is $32.40. How much does Item A cost?

2. Tom has some money. Bruce and Dave each have $1 more than Tom, and Paul, Dan and Neil each have twice as much as Tom. Together the six boys have $15.50. How much does each have?

3. A man has three children who each want a calculator for Christmas. He buys the two older children the same model and the youngest child gets a basic model costing $20 less. What is the value of each type of calculator if the man spends a total of $86.50 on the calculators?

4. The perimeter of a room is 50m. The length exceeds the width by 3.3m. Find the length of the room.

5. Four consecutive natural numbers have a sum of 110. What is the smallest of these numbers?

6. A girl purchases a number of items each costing $17.35. Her bank account was $1000.00 before she made the purchase. Afterwards it was $687.70. How many did she purchase?

7. Tina and Jane purchase raffle tickets and agree to share the money in the ratio of how much each paid. They win $117.00 and Jane, who put in $2.00, gets $26.00. What was Tina's share of the cost of the raffle tickets?

8. The distance from B to C is $\frac{7}{11}$ of the distance from A to B. The distance from A to B exceeds that from B to C by 20km. How far is it from A to B?

9. A man purchases a number of tools at $5 each. Unfortunately, five do not work but by selling those that do at $12 each he is able to make $45 profit. How many tools did he buy originally?

10. All prices are to rise by 15% as a result of a new tax. A shopkeeper had increased the cost of a particular item by $5.00 before the new tax came into effect. As a result of the second price rise, this item now costs $13.80. What did it cost originally?

11. Inflation is running at a certain percentage. After one year an item which cost $36.00 costs $40.86. What is the inflation rate?

12. The denominator of a fraction is 44 more than the numerator. The fraction, when simplified, is $\frac{13}{17}$. Find the numerator.

13. My son is 31 years younger than me. In one year's time my son's age will be one quarter of my current age. How old am I?

14. One girl has four times as much money as her friend. She gives her friend $12.00 and as a result they now have the same amount of money. How much did each have originally?

15. A husband and his wife decide to invest in a business. The first time they do this he invests $3 000.00 and she invests $5 000.00. The next time they invest, they each contribute the same amount. After the two investments are made the ratio of the husband's investment to his wife's is 13:17. How much did each invest the second time?

Inequations

Inequations are mathematical sentences in which one of the following inequality symbols is used (instead of an equals sign as in an equation).

symbol	symbol means	example
$>$	is greater than	$5 > 1$, $y > -26$
$\geq$	is greater than or equal to	$8 \geq x$, $45 \geq 45$
$<$	is less than	$1 < 3$, $-2 < -1$, $a < b + 1$
$\leq$	is less than or equal to	$5 \leq 5$, $a + b \leq 25.7$
$\neq$	is not equal to	$3 \neq 2$
$\ngtr$	is not greater than	$2 \ngtr 4$, $3 \ngtr 3$
$\ngeq$	is not greater than or equal to	$2 \ngeq 5$
$\nless$	is not less than	$4 \nless 3$, $2 \nless 2$
$\nleq$	is not less than or equal to	$5 \nleq 4$
$\neq$	is the same as $<$ or $>$.	
$\nless$	is the same as $\geq$.	
$\nleq$	is the same as $>$.	
$\ngtr$	is the same as $\leq$.	
$\ngeq$	is the same as $<$.	

Solving an inequation means finding the values which make the inequation true. The technique is similar to that of solving an equation except that the *inequality sign is reversed* when multiplying or dividing by a *negative* number.

Example K: Solve $5 - 3x \le 8$

Solution: $5 - 3x \le 8$

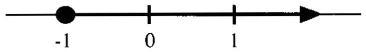
$5 + -3x \le 8$

$-3x \le 3$ [subtracting 5]

$x \ge \frac{3}{-3}$ [inequality *reverses* when dividing by -3]

$x \ge -1$

On a number line the **solution set** appears as:



Exercise 2g

For each of the following inequations find the solution set. Answers should be given as an inequality.

1. $3x > 5$
2. $2x - 1 \le 4$
3. $5 - 3x \ge -4$
4. $\frac{2 - 3x}{4} < 2$
5. $\frac{x+3}{2} - \frac{x-1}{3} \ge 1$
6. $\frac{x+3}{2} < \frac{x-1}{3} - 1$
7. $1\frac{1}{3}(x - 2) - \frac{1}{2}(x + 3) \ge x + 3$
8. $\frac{a}{5} - \frac{2}{3}(1 - a) \ge \frac{a+1}{2}$
9. $\frac{2x+3}{3} < \frac{3x-1}{2}$
10. $\frac{4x-1}{3} \ge \frac{x+1}{7} - 1$

11. Anyone wishing to keep animals of a certain breed must pay an annual licence fee of $45 irrespective of how many they keep. There is an annual cost per animal of $167.50 covering feeding, etc. The total cost of keeping these animals must always be less than $1070 per year.
    a. Write an inequation for the number of animals kept per year.
    b. Find the largest number which can be kept.

12. The weight of an empty barrow is 20 kg. The barrow contains bricks with an average weight of 1.3 kg. The total weight of a barrow containing bricks must exceed 108 kg.
    a. Write an inequation for the number of bricks in the barrow.
    b. Find the minimum number of bricks the barrow can hold.

13. The weight of an empty box must be less than 0.8 kg. Apples weigh 0.21 kg on average. A full box weighs 17.2 kg.
    a. Write an inequation for the number of apples in the box.
    b. Find the minimum number of apples in the box.

14. Danny and Denise went shopping. Danny had $50, Denise had $60. Danny has to bank one third of what he didn't spend. Denise has to bank two fifths of what she didn't spend. Their combined banking has to exceed $15.
    a. Choose appropriate letters and write an inequation summarising the above.
    b. If Denise didn't spend $22, find the largest amount Danny could have spent.

15. A farmer milks all except 3 of her cows and 7 of her goats. She gets checked by MAF who find $\frac{1}{4}$ of her milked cows have an impurity in their milk, and $\frac{1}{6}$ of her milked goats also. If the number of animals with impurities is less than 15 then MAF will not fine her.
    a. Write an inequation summarising the above.
    b. If she runs 23 goats find the maximum number of cows she could run and not be fined.

Problems and Investigations

1. A rental car company runs small cars and large cars. It must run at least 4 of each type. The total number of cars must never exceed 20. The number of small cars must never exceed the number of large cars by more than 3. The profit per day per small car is $25 and per large car is $35. Find the maximum profit per day.

2. A dealer in vehicles sells bicycles, cars and tricycles. He always has some of each in stock. A child counts the number of wheels he saw and told his mother that he counted 150 (ignore spare tyres, etc). Investigate this and find the largest possible number of each type of vehicle there could have been in the showroom.

3. EXPANDING AND FACTORISING

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

- LEVEL 5 ALGEBRA
  - *to factorise and expand algebraic expressions*
- LEVEL 7 ALGEBRA
  - *to carry out appropriate manipulation and simplification of algebraic expressions*

Expanding

Brackets are removed from and inserted into algebraic expressions using the **distributive law**. The distributive law states that if a, b and c are numbers then:

$$\begin{aligned} a(b + c) &= a \times b + a \times c \\ &= ab + ac \end{aligned}$$

Expanding an expression means removing brackets by multiplying using the distributive law. Where possible, the expression is then simplified.

Example A:

a. $5x(3 - 2x) = 5x \times 3 - 5x \times 2x$ [expanding by using the distribution law]
$= 15x - 10x^2$

b. $2x^2(3x^3 + 2x) = 2x^2 \times 3x^3 + 2x^2 \times 2x$ [using the distributive law]
$= 6x^5 + 4x^3$

c. $(2x + 3)(4x + 5) = (2x + 3)4x + (2x + 3)5$ [using the distributive law]
$= 4x(2x + 3) + 5(2x + 3)$ [multiplication is commutative]
$= 8x^2 + 12x + 10x + 15$ [expanding]
$= 8x^2 + 22x + 15$ [simplifying]

Note: $(2x + 3)(4x + 5)$ is expanded by multiplying every possible pair from the brackets [indicated by arrows] then simplifying.

$$\begin{aligned} (2x + 3)(4x + 5) &= 8x^2 + 10x + 12x + 15 \\ &= 8x^2 + 22x + 15 \end{aligned}$$

d. $(2x - 1)(x + 3)^2 = (2x - 1)(x + 3)(x + 3)$ [expanding $(x + 3)^2$]
$= (2x + {}^{-}1)(x^2 + 3x + 3x + 9)$ [$2x - 1$ changed to $2x + {}^{-}1$]
$= (2x + {}^{-}1)(x^2 + 6x + 9)$ [simplifying]
$= 2x^3 + 12x^2 + 18x + {}^{-}x^2 + {}^{-}6x + {}^{-}9$ [expanding brackets]
$= 2x^3 + 11x^2 + 12x - 9$ [simplifying]

Exercise 3a: Expanding

Expand these expressions:

1. $5(3x + 1)$
2. $5x(2x + 3)$
3. $(x + y)(2x + y)$
4. $(2x + 1)(3x + 4)$
5. $(3A - 2)(4A + 5)$
6. $(2x + 7)(x - 3)$
7. $(6P - 8)(2Q + 3)$
8. $(2A + B)(3A - B)$
9. $(x - y)(3x - y + 2)$
10. $(x + 2y)(3x + 2y - 3)$
11. $(2x - y + 4)(x - y + 2)$
12. $(x + 1)^2$
13. $(3x - 4y)^2$
14. $(P + R - 2)^2$
15. $(A + B)^3$
16. $(x + 2)^2(x - 2)^2$
17. $((x + 1) - (x - 1))^3$
18. $(3A^2 + 2BA + B^2)(4A^2 - 3AB + 5B^2)$
19. $(x^2 - 2x + 3)^2$
20. $(2x + 3y)^3$

Example B: Expand and simplify $4(2x + y - 2) - 3(x - y - 3)$.

Solution:

$4(2x + y - 2) - 3(x - y - 3) = 8x + 4y - 8 - 3(x - y - 3)$ [expanding first bracket]
$= 8x + 4y - 8 - 3x + 3y + 9$ [expanding second bracket]
$= 5x + 7y + 1$ [simplifying]

Exercise 3b: Expanding and Simplifying

Simplify the following expressions as much as possible:

1. $5(x + 1) + 2(x + 3)$
2. $4(A + B) + 3(A - B)$
3. $6(x + y) - 3y$
4. $6(x + 2y + 3) + 2(x - 2y - 2)$
5. $7(2x + 3y) - 2(x + y)$
6. $5(A - 3B) - 2(B + A)$
7. $x^2 + 3x + 2(x - 1)$
8. $x(2x + 3) - 3(x + 4)$
9. $3x^2 + 4x + 5 - x(x - 2)$
10. $5x - (3x - 7)$
11. $7(x + 1)^2 - (2x + 3)^2(2x^2 - 3x + 2)$
12. $(2A + 1)^3 - 2(A^3 - 2A^2 + 3A + 1)$
13. $(2A + 3)^3 - (2A - 1)^3$
14. $x^2(x + 1) - (x - 1)^3$

Factorising

Factorising an algebraic expression means writing the expression as a product of factors which are simpler algebraic expressions. This is done by first looking for **common factors** of the terms in the expression.

A common factor is a number or expression which divides each term without remainder.

Example C:

a. $6x - 12$ can be written $6 \times x - 6 \times 2$ [6 is a common factor]
Thus $6x - 12$ can be factorised to give $6(x - 2)$.

b. $2A^2 + 7A = A \times 2A + A \times 7$ [A is a common factor]
$= A(2A + 7)$

c. $5L^3m^2n^2 - 10L^2m^3n^2 + 15L^2m^2n^3 = 5L^2m^2n^2(L - 2m + 3n)$
[$5L^2m^2n^2$ is a common factor]

Example D: Factorise: $AP - AQ + 2Q - 2P$

Solution:

$AP - AQ + 2Q - 2P = A(P - Q) + 2(Q - P)$ [factorising AP – AQ and 2Q – 2P separately]
$= A(P - Q) - 2(P - Q)$ [$-(P - Q) = Q - P$]
$= (A - 2)(P - Q)$ [$(P - Q)$ is a common factor]

Factorising Quadratics

An algebraic expression of the form $Ax^2 + Bx + C$, where A, B and C are numbers, is called a **quadratic expression** or just a **quadratic**.

In the simplest case, when the coefficient of x^2 is 1, the quadratic simplifies to $x^2 + Bx + C$. Such expressions are factorised by finding two numbers which *multiply together to give C and add together to give B.*

Example E:

a. To factorise $x^2 + 7x + 12$, look for two numbers which add to give 7 and multiply together to give 12. The numbers are 3 and 4 since $3 + 4 = 7$ and $3 \times 4 = 12$. Thus $x^2 + 7x + 12 = (x + 4)(x + 3)$.

b. Factorise $A^2 - 7A - 18 = (A - 9)(A + 2)$ [since $-9 + 2 = -7$ and $-9 \times 2 = -18$]

If A is not 1, the quadratic to be factorised is $Ax^2 + Bx + C$. To find the factors 2×2 **matrices** are set up, in which the first column has pairs of numbers which multiply to give A and the second column has pairs of numbers which multiply to give C. When the diagonal products of the matrices add to give B, the pairs of numbers which will factorise the equation have been found.

Example F: Factorise $4B^2 + 4B - 15$

Solution: Set up some 2×2 matrices in which the first column has a pair of numbers which multiply to give 4 and the 2nd column has a pair of numbers which multiply to give $^-15$. Look for the matrix where the sum of the diagonal products is 4.

Matrices	diagonal gives:	
$\begin{pmatrix} 2 & -15 \\ 2 & 1 \end{pmatrix}$	$2 \times 1 + 2 \times -15 = -28$	[factors incorrect]
$\begin{pmatrix} 2 & -5 \\ 2 & 3 \end{pmatrix}$	$2 \times 3 + 2 \times -5 = -4$	[factors incorrect]
$\begin{pmatrix} 2 & 5 \\ 2 & -3 \end{pmatrix}$	$2 \times -3 + 2 \times 5 = 4$	[factors correct]

$\therefore\ 4B^2 + 4B - 15 = (2B + 5)(2B - 3)$ [reading matrix horizontally]

With practice most factorisations can be done mentally.

Example G: $6x^2 - 13x - 28 = (2x - 7)(3x + 4)$

$\begin{pmatrix} 3 & -7 \\ 2 & 4 \end{pmatrix}$ $12 + -14 = -2$ [factors incorrect]

and $\begin{pmatrix} 2 & -7 \\ 3 & 4 \end{pmatrix} \rightarrow 8 + -21 = -13$ [factors correct]

The Difference of Two Squares

The **difference of two squares** $x^2 - y^2$ has a particularly important factorisation which is:

$$x^2 - y^2 = (x - y)(x + y)$$

This can be confirmed by expanding the right-hand side. Expressions which can be written as the difference of two squares are easily factorised.

Example H: $49x^2 - 36B^2 = (7x)^2 - (6B)^2$ [writing $49x^2$ and $36B^2$ as 'squares']
$= (7x - 6B)(7x + 6B)$

Exercise 3c: Factorising

Factorise the following: (express with brackets)

1. $ax - ab$	2. $ab - a^2$	3. $11x^2 - x$
4. $4ax + 8ay$	5. $x^2 + 7x + 12$	6. $x^2 + 13x + 36$
7. $P^2 + 12P + 27$	8. $P^2 - 3PQ + 2Q^2$	9. $R^2 - 6RP + 8P^2$
10. $x^2 - xy - 42y^2$	11. $P^2 + 4PQ - 5Q^2$	12. $P^2 - 3PQ - 10Q^2$
13. $R^2 + 10RS + 24S^2$	14. $x^2 - 49$	15. $P^2 - 36P$
16. $L^2 - 5L + 6$	17. $M^2 + 16M + 60$	18. $M^2 - 7M - 8$
19. $25M^2 - 100$	20. $2x^2 + 3x + 1$	21. $x^2 - 6xb + 5b^2$
22. $2x^2 + 7x + 3$	23. $4P^2 + 21P + 5$	24. $3z^2 + 4z - 7$
25. $4P^2 + 5P - 21$	26. $12x^2 - 11x + 2$	27. $L^2P^2 - 2LP - 24$
28. $14 + 11x - 15x^2$	29. $7p^2 + 6pq - q^2$	30. $8L^2 - 32x^2$
31. $27Px^2 - 3Py^2$	32. $50 - 2x^2$	33. $3AP^2 - 4AP + A$
34. $9x^2 + x^7$	35. $18xy - 21x^2 - 14x$	36. $8p^3q^3 + 16p^4q^3 + 24p^5q^4$
37. $p^6q^4r^4 - 9x^{10}y^{12}$	38. $x^5 - b^2x^3 + b^3x^2 - b^5$	39. $A(C - D) + B(D - C)$
40. $x^2 + 2x + 1 + ax + a$		

The Sum and Difference of Two Cubes

The **difference of two cubes** $(x^3 - y^3)$ and their **sum**, $(x^3 + y^3)$, can be written:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$
$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

This can be confirmed in each case by expanding the right-hand side.

Example I:

a. $x^3 - 1 = x^3 - 1^3$ [writing $x^3 - 1$ as a difference of two cubes]
$= (x - 1)(x^2 + 1x + 1^2)$ [since $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$]
$= (x - 1)(x^2 + x + 1)$

b. $A^3 + 8 = A^3 + 2^3$ [writing $A^3 + 8$ as a sum of two cubes]
$= (A + 2)(A^2 - 2 \times A + 2^2)$ [since $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$]
$= (A + 2)(A^2 - 2A + 4)$ [simplifying]

Exercise 3d: More Difficult Factorisation

Factorise the following:

1. $1 + x^3$	2. $z^3 - 8$	3. $27 - z^3$
4. $64x^3 - 27k^3$	5. $m^6 - n^6$	6. $Ax + Ay - Bx - By$
7. $m^2n - n^3 + (m + n)^2$	8. $(x + 3)^2 - 2(x + 3) - 8$	9. $(x + y)^2 - (a + b)^2$
10. $ab(z^2 + 1) - z(a^2 + b^2)$		

Simplifying Quotients

Quotients are expressions which have a *denominator* and a *numerator*. They are easily simplified if the denominator and numerator can be factorised to give a common factor.

Example J: $\dfrac{x^2 - 8x + 12}{x^2 - 36} = \dfrac{(x - 6)(x - 2)}{(x - 6)(x + 6)}$ [factorising numerator and denominator]

$= \dfrac{x - 2}{x + 6}$ [cancelling the common factor, $(x - 6)$]

Note: Cancelling can only be done if the numerator and denominator are both expressed as products which have common factors.

Exercise 3e: Simplifying Quotients

Simplify the following expressions as much as possible:

1. $\dfrac{x^2 - 4x - 5}{(x + 1)^2}$	2. $\dfrac{x^2 + 4x + 4}{x^2 + 5x + 6}$	3. $\dfrac{x^2 + 5x}{x^2 + 6x + 5}$
4. $\dfrac{x^2 - 49}{x^2 - 5x - 14}$	5. $\dfrac{(x + 1)^2}{(x + 1)^3}$	6. $\dfrac{x^2 + 9x + 20}{x^2 - 16}$
7. $\dfrac{p^2 + 11p + 28}{p^2 + 7p + 12}$	8. $\dfrac{5p + 15}{p^2 + 4p + 3}$	9. $\dfrac{a^2 - ab}{ab}$
10. $\dfrac{3y^2 - 27}{12y^2 + 36y}$	11. $\dfrac{2y^2 + 3y + 1}{y^2 + 2y + 1}$	12. $\dfrac{a^2 - (b - c)^2}{b^2 - (a - c)^2}$
13. $\dfrac{a^3 + 1}{(a + 1)^2}$	14. $\dfrac{x^3 + y^3}{x^2z - xyz + y^2z}$	15. $\dfrac{x^2 + ax + bx + ab}{x^2 + 2ax + a^2}$

Problems and Investigations

1. Here are some mathematical expressions which are equivalent to $8x + 5$.
   i. $4x + 4x + 5$ ii. $8(x + 1) - 3$

   (If two mathematical expressions are equivalent then they always have the same value no matter what value the variable takes.)

   a. Write ten other expressions equivalent to $8x + 5$.
   b. i. Find a mathematical expression equivalent to $8x + 5$ which can be written in the form $A(x + 1) + Bx + C$ where A, B, C are natural numbers (positive integers).
      ii. What is the total number of such possible equivalent expressions to $8x + 5$?

2. a. Investigate the following pattern of sums.
      $1^3 + 2^3$, $1^3 + 2^3 + 3^3$, $1^3 + 2^3 + 3^3 + 4^3, \ldots$
   b. Find a formula for the sum of n consecutive cubes.
   c. How many cubes would you have to add to exceed 1 000 000?
   d. Prove that the sum of consecutive cubes is the difference of two squares.

4. RATIONAL EXPRESSIONS

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 5 ALGEBRA

- *to simplify algebraic fractions*
- *to factorise and expand algebraic expressions*

LEVEL 7 ALGEBRA

- *to carry out appropriate manipulation and simplification of algebraic expressions*

Rational expressions are sometimes called **algebraic fractions**. They are fractions involving letters as well as numbers. The procedures used to simplify rational expressions are similar to those used with fractions involving only numbers.

Multiplication

Example A: Express $\frac{x}{y} \times \frac{a}{c}$ as a single fraction.

Solution: $\frac{x}{y} \times \frac{a}{c} = \frac{x\,a}{y\,c}$ [numerators and denominators are multiplied together]

Example B: Simplify $\frac{x}{y} \times \frac{y}{z} \times \frac{z^2}{x^3}$

Solution:

$$\frac{x}{y} \times \frac{y}{z} \times \frac{z^2}{x^3} = \frac{xyz^2}{x^3yz}$$

$$= \frac{x.y.z.z}{x.x^2.y.z} \quad \text{[writing } z^2 = z\,.\,z \text{ (see Ex. N, Chapter 1)]} \quad x^3 = x\,.\,x^2$$

$$= \frac{z}{x^2}$$

Some expressions are best factorised *before* any multiplication is done:

Example C: $\dfrac{x^2+3x+2}{(x+2)^2} \times \dfrac{5}{x+1} = \dfrac{(x+1)(x+2)}{(x+2)(x+2)} \times \dfrac{5}{(x+1)}$ [factorising]

$= \dfrac{5}{x+2}$ [cancelling $(x + 1)$ and $(x + 2)$]

Exercise 4a

1. Simplify each of the following:

a. $\frac{3}{4} \times \frac{a}{b}$ b. $\frac{5a}{b} \times \frac{c}{d}$ c. $\left(\frac{3}{4} \times \frac{a}{d}\right) \times \frac{c}{b}$

d. $\frac{a}{b} \times \frac{a}{c}$ e. $\frac{3}{a} \times \frac{ab}{6}$ f. $\frac{pq}{r} \times \frac{3r}{q}$

g. $\frac{5a^2}{3} \times \frac{9}{a}$ h. $\left(\frac{5}{a} \times \frac{a^2}{10}\right) \times \frac{1}{3}$ i. $\frac{6k}{3q} \times \frac{qk}{3}$

j. $3\frac{1}{4} \times \frac{b}{2}$ k. $\frac{x^2}{y} \times \frac{y^3}{x^4} \times \frac{y}{x}$ l. $\frac{2x^2}{y} \times \frac{xy^2}{6x^4}$

2. Simplify each of the following:

a. $\dfrac{x+1}{x+3} \times \dfrac{x^2+3x}{x^2+2x+1}$ b. $\dfrac{x}{x^2-1} \times \dfrac{x^2+4x+3}{x^2+3x}$

c. $\dfrac{(x+3)^2}{x^2+5x+6} \times \dfrac{x^2-9x+14}{x^2-4x-21}$ d. $\dfrac{4x+8}{x^2+11x+28} \times \dfrac{x^2+9x+14}{x^2+4x+4}$

Division

Division means multiplication by the *reciprocal* of the **divisor**.

Example D: Simplify $\frac{x^2}{3} \div \frac{x}{9}$

Solution: $\dfrac{x^2}{3} \div \dfrac{x}{9} = \dfrac{x^2}{3} \times \dfrac{9}{x}$ [multiplying by the reciprocal]

$= \dfrac{9x^2}{3x}$

$= 3x$

Example E:

$\dfrac{x^2-y^2}{x^2+2xy+y^2} \div \dfrac{ax-ay}{2x+2y} = \dfrac{x^2-y^2}{x^2+2xy+y^2} \times \dfrac{2x+2y}{ax-ay}$ [multiplication by reciprocal]

$= \dfrac{(x-y)(x+y)}{(x+y)(x+y)} \times \dfrac{2(x+y)}{a(x-y)}$ [factorising]

$= \dfrac{2}{a}$ [cancelling]

Exercise 4b

1. Simplify each of the following:

a. $\frac{2}{3} \div \frac{a}{c}$ b. $\frac{ab}{c} \div \frac{d}{e}$ c. $\frac{x^2}{y} \div \frac{y}{3}$

d. $\left(\frac{2}{5} \times \frac{a}{3}\right) \div \frac{2}{3}$ e. $\left(\frac{a}{c} \times \frac{5}{3}\right) \times \left(\frac{a}{b} \times \frac{5}{3}\right)$ f. $\frac{ab}{3} \div \frac{a^2}{9}$

g. $\frac{4a^2b}{3} \div \frac{2ab}{3}$

2. Simplify each expression to a single fraction.

a. $\frac{2}{3} \div \frac{5}{a+1}$ b. $\frac{5}{7} \div \frac{x}{x^2-1}$ c. $\frac{b-1}{3} \div \frac{3}{4}$

d. $\frac{2}{3} \div \left(\frac{y}{x+y}\right)$ e. $\frac{3x-1}{2} \div \frac{2}{3}$ f. $\frac{3x}{x+1} \div 2\frac{1}{2}$

3. Simplify each expression.

a. $\dfrac{5x-10}{x+3} \div \dfrac{x-2}{x+3}$ b. $\dfrac{x+4}{x-1} \div \dfrac{x+4}{x+2}$ c. $\dfrac{x^2+x}{x+2} \div \dfrac{x}{x^2-4}$

d. $\dfrac{x+3}{x+7} \div \dfrac{x^2+4x+3}{x^2+10x+21}$ e. $\dfrac{x^2+2x+1}{x^2+3x+2} \div \dfrac{x^2+6x+5}{x^2-4}$

Addition and Subtraction

Addition and subtraction of any fraction involves changing the fractions to equivalent fractions so that they have a **common denominator**.

Example F: $\dfrac{x}{3} + \dfrac{y}{4} = \dfrac{4x}{12} + \dfrac{3y}{12}$ [changing both fractions to equivalent fractions with common denominator]

$= \dfrac{4x+3y}{12}$

Example G: Express $\frac{a}{bx} + \frac{y}{x^2} - \frac{2}{b^2}$ as a single fraction.

Solution:

$\dfrac{a}{bx} + \dfrac{y}{x^2} - \dfrac{2}{b^2} = \dfrac{abx}{b^2x^2} + \dfrac{yb^2}{b^2x^2} - \dfrac{2x^2}{b^2x^2}$ [changing to equivalent fractions with common denominator of b^2x^2]

$= \dfrac{abx + yb^2 - 2x^2}{b^2x^2}$

Example H: $\frac{a+3}{2} - \frac{a-3}{3}$

$= \frac{3a+9}{6} - \frac{2a-6}{6}$ [changing to a common denominator of 6]

$= \frac{(3a+9)-(2a-6)}{6}$ [using brackets to avoid errors]

$= \frac{3a+9-2a+6}{6}$ [removing brackets]

$= \frac{a+15}{6}$ [simplifying]

Example I: $\frac{3}{x+y} - \frac{2}{x-y} + \frac{1}{x^2-y^2}$

$= \frac{3}{x+y} - \frac{2}{x-y} + \frac{1}{(x-y)(x+y)}$ [factorising the denominator]

$= \frac{3(x-y)}{(x+y)(x-y)} - \frac{2(x+y)}{(x+y)(x-y)} + \frac{1}{(x+y)(x-y)}$ [changing to a common denominator]

$= \frac{3(x-y)-2(x+y)+1}{(x-y)(x+y)}$

$= \frac{3x-3y-2x-2y+1}{(x-y)(x+y)}$ [removing brackets]

$= \frac{x-5y+1}{(x-y)(x+y)}$ [simplifying]

Use of Calculators

Very few calculators, even the most advanced, are of much help in tackling this topic. There is a computer program called 'Mathematica' which enables the easy manipulation and simplification of the expressions found in this chapter.

Exercise 4c

1. Express each of the following as simply as possible:

a. $\frac{a}{b} + \frac{b}{3}$ b. $\frac{a}{8} + \frac{a}{4}$ c. $\frac{a}{b} + \frac{1}{b}$ d. $\frac{a}{b} + \frac{1}{4}$

e. $\frac{a}{3} - \frac{b}{4}$ f. $\frac{a}{c} - \frac{b}{a}$ g. $\frac{1}{a} + \frac{2}{a^2}$ h. $\frac{2}{3} \times \frac{a}{b} + \frac{1}{b}$

i. $2\frac{1}{2} + \frac{b}{4}$ j. $3p + \frac{2q}{7}$

2. Simplify each of these expressions to a single fraction:

a. $\frac{a+3}{2} + \frac{a}{2}$ b. $\frac{2b+3a}{3} + \frac{a+b}{3}$ c. $\frac{a+b}{2} + \frac{2a+7b}{3}$

d. $a + \frac{a+4}{3}$ e. $\frac{5b+11}{2} - \frac{1}{3}$ f. $\frac{4a+1}{a} + \frac{1}{2}$

3. Simplify these expressions to a single fraction:

a. $\frac{a+1}{3} - \frac{a-1}{4}$ b. $\frac{2b+3}{3} - \frac{3-b}{5}$ c. $\frac{p-q}{4} - \frac{p+q}{3}$

d. $\frac{3p}{4} - \frac{p-2q}{5}$ e. $\frac{a}{3} - \frac{4-3a}{5}$ f. $\frac{7-p}{2} - \frac{p}{3}$

g. $\frac{18}{5} - \frac{2-p}{2}$ h. $\frac{3+r}{4} - \frac{2-r}{3}$ i. $\frac{6p-q}{3} - q$

4. Simplify each of these expressions to a single fraction:

a. $\frac{1}{x+1} + \frac{1}{x+2}$ b. $\frac{2}{p+3} + \frac{1}{p+1}$ c. $\frac{2}{p+1} + \frac{1}{p-3}$

d. $\frac{3}{R+4} + \frac{2}{R-2}$ e. $\frac{3}{R} - \frac{1}{R+3}$ f. $\frac{1}{A+3} - \frac{1}{R+2}$

g. $\frac{4}{A-3} - \frac{1}{A-1}$ h. $\frac{A+2}{A+1} + \frac{A}{A-1}$ i. $\frac{x}{2(x+1)} - \frac{1}{3(x+1)}$

j. $\frac{y}{3(x+y)} - \frac{x}{4(x-y)}$ k. $\frac{2}{(x+1)(x+2)} + \frac{3}{(x+2)(x+3)}$

l. $\frac{4}{x(x+1)} - \frac{3}{(x+1)(x+2)}$ m. $\frac{1}{x-y+2} + \frac{2}{x+y-3}$

n. $\frac{3}{x(x+y-2)} - \frac{2}{x(x-y+2)}$ o. $\frac{4}{x(x-y)} - \frac{3}{y(x-y)}$

5. Express each of the following as single fractions, simplifying where possible:

a. $\frac{2ab}{10a^2b^3} \times \frac{25a^3b}{15ab^2}$ b. $\frac{6xyz^2}{15x^2z} \times \frac{10y^3z^3}{9xy^2}$

c. $\frac{9a^2c}{a-c} \times \frac{2a^2-2c}{6ac^3}$ d. $\frac{x^2+4x+4}{x^2-9} \times \frac{x^2+5x+6}{x(x+2)^2}$

e. $\frac{u^2-36}{x^2+5x+4} + \frac{u^2+5u-6}{x^2-1}$ f. $\frac{2a^2-98}{a^2-25} \times \frac{a^2-3a-10}{3a-21} + \frac{a^2+5a-14}{2a+10}$

g. $\frac{2}{x-y} + \frac{1}{2x-2y}$ h. $\frac{2}{x+1} + \frac{3}{x-1}$

i. $\frac{2}{x-2y} - \frac{1}{x+2y}$ j. $\frac{1}{x+y} + \frac{y}{x^2-y^2}$

k. $\frac{1}{x+1} + \frac{1}{x+2} - \frac{3}{x^2+3x+2}$ l. $\frac{1}{a-4} - \frac{2}{a-5} - \frac{1}{a-6}$

m. $\frac{1}{x^2-1} + \frac{1}{x^2+2x-3} + \frac{2}{x^2+4x+3}$ n. $\frac{a(a-1)(a-2)}{4} + \frac{(a-1)(a-2)(a-3)}{3}$

Problems and Investigations

1. Notice that $\frac{3A+3}{4}$ can be calculated in a variety of ways.

   Examples: i. $\frac{3}{4} \times (A+1)$

   ii. $(A+1) \div \frac{4}{3}$

   iii. $3 \times \frac{A+1}{4}$

   Write down as many calculations as you can which give each of the following algebraic expressions.

   a. $\frac{5A-5}{7}$ b. $\frac{x^2+x}{x^2-4}$ c. $\frac{x^2}{(x+1)(x^2+5x+4)}$

2. a. Investigate the following sequences of fraction subtractions.

   i. $\frac{1}{2}-\frac{1}{3}, \frac{1}{2}-\frac{1}{5}, \frac{1}{2}-\frac{1}{7}, \ldots$

   ii. $\frac{1}{5}-\frac{1}{6}, \frac{1}{5}-\frac{1}{7}, \frac{1}{5}-\frac{1}{8}, \ldots$

   b. After examining the results of 2a, express each of the following fractions as the difference of the reciprocals of two integers.

   i. $\frac{1}{20}, \frac{5}{36}, \frac{3}{40}, \frac{7}{18}, \frac{6}{55}, \frac{9}{36}$

   c. Prove any fraction can be written as the difference of other fractions.

   d. Express the fraction $\frac{A}{BC}$ as the difference of 2 fractions $\frac{X}{Y}$ and $\frac{Z}{W}$ where each of X, Y, Z, W are algebraic expressions involving sums, products or differences of A, B, C.

5. FORMULAE

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 4 ALGEBRA

- *to find and justify a word formula which represents a given practical situation*

LEVEL 5 ALGEBRA

- *to use equations to represent practical situations*

LEVEL 6 ALGEBRA

- *to substitute values into formulae*

LEVEL 7 ALGEBRA

- *to write appropriate equations or inequations to describe practical situations*

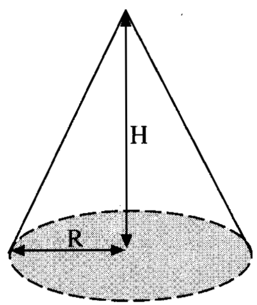
Introduction

Mathematical modelling involves expressing a relationship as a **formula** in which mathematical symbols and letters are used to represent **variables**. The ability to do this accurately and use the resulting formula is important in many situations.

Example A: Three quantities used in accounting are assets, A, proprietorship, P, and liabilities, L. The relationship between these quantities is that assets are equal to the sum of proprietorship and liabilities. In mathematical form, the relationship can be expressed: A = P + L.

Note: A, P and L are called **terms** and the equation, A = P + L is a formula for A in terms of the two variables, P and L.

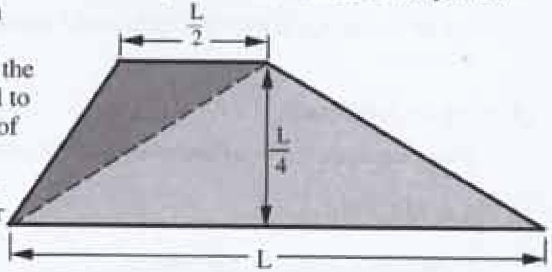
Example B: The volume of a cone is one third of the product of its height and the area of its base. Express this relationship in a mathematical form.



Solution: Let the volume of the cone be V, the radius of the base R, and the height, H. The base area is πR^2.

$$\therefore \text{ the volume is } V = \frac{1}{3}\pi R^2 H$$

Example C: A trapezium is a quadrilateral with a pair of parallel sides. Find an expression for the area of the trapezium shown where one parallel side is one half the length of the other parallel side, and the distance between the two parallel sides is equal to one quarter of the length of the longer parallel side. Express the area in terms of the length of the longer parallel side.



Solution: Let the length of the longer parallel side be L and the area be A. Area of the trapezium is the sum of the areas of the two shaded triangles.

area of the smaller triangle is $\frac{1}{2} \times \frac{L}{2} \times \frac{L}{4} = \frac{L^2}{16}$ [area is half × base × height]

area of the larger triangle is $\frac{1}{2} \times L \times \frac{L}{4} = \frac{L^2}{8}$

$\therefore$ area of the trapezium is $A = \frac{L^2}{16} + \frac{L^2}{8} = \frac{3L^2}{16}$

Exercise 5a: Writing Formulae

1. A pair of socks costs \$D. Write an expression for the cost of the socks in cents.

2. a. Express the time of $3x$ minutes in hours.
   b. Express this same time in seconds.

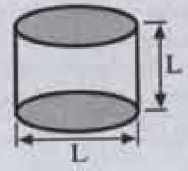
3. Peter weighs x kilograms, y grams.
   a. Express Peter's weight in grams. b. Express his weight in kilograms

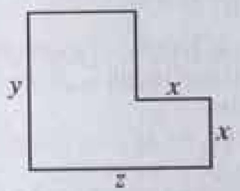
4. A rectangle has length L metres and width W centimetres. Write an expression for:
   a. the area in cm^2. b. the area in m^2. c. the perimeter in mm.

5. A square has area $4L^2$ in square metres. Write an expression for:
   a. the perimeter in cm. b. the perimeter in mm.

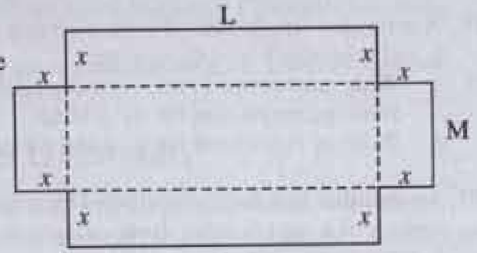
6. A right-angled triangle has base b and height a. Both have the same unit.
   a. What is the area of the triangle?
   b. What is the length of the **hypotenuse** [the third side]?

7. a. Find an expression for the volume of the cylinder shown.
   b. Find an expression for the total surface area of the cylinder.



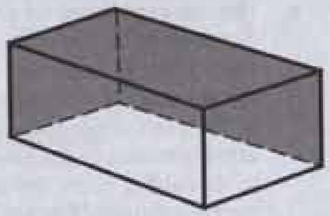


8. a. Find an expression for the perimeter of the shape shown.
   b. Find a formula for the area of this shape.



9. a. What is the perimeter of the shape shown?
   b. What is the volume of the box which forms when the shape is folded along the dotted lines?

10. The length of the closed box shown is 4L. The height and width of the box are both half the length.
    a. Write down an expression for the volume of the box.
    b. Write down an expression for the surface area of the box.



11. Find the formula for the perimeter of an **isosceles triangle** in which the third side of length L is one half the length of each of the other sides.

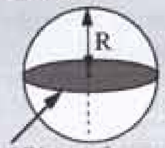
12. Find the formula for the area of a rectangle in terms of its length if its length is always double its width.

13. Find the formula for the perimeter of the rectangle described in question 12. Give your answer in terms of the length.

14. Find the formula for the circumference of a semicircular disc in terms of its radius.

15. Find the formula for the volume of a box in terms of its height, H, if its height always exceeds its length by 1m and is double its width.

16. The volume of a sphere is the product of four thirds of its radius times the area of its internal plane of symmetry. Express the volume in terms of its radius.



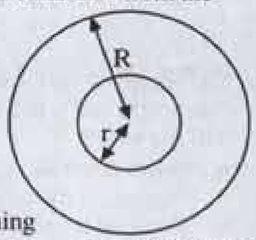
Internal Plane of symmetry

17. A cylinder has base radius R and height H.
   a. Find an expression for its volume.
   b. Find how many such cylinders full of water would be needed to fill another cylinder whose radius was three times as large and whose height was twice as great.

18. A shape is made by welding the base of a rectangle of base b and length l onto the side of a much larger rectangle which has base B and length L.
   a. Find an expression for the perimeter of this shape.
   b. What is the effect on the perimeter of doubling b?

19. A prism has length 10L. Its cross-section is a right-angled triangle of height L and base equal to half its height.
   a. Write an expression for its cross-sectional area.
   b. Write an expression for its volume.
   c. Write an expression for its surface area.

20. An **annulus** is a shape constructed by removing a circular disc from the interior of a larger circle. Both circles have the same centre.



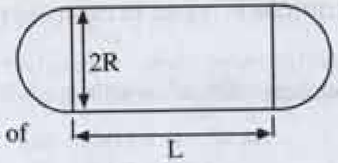
   a. Find an expression for the area of an annulus constructed by removing a disc of radius r from a disc of radius R.
   b. An ant starts at one point on the circumference of the inner circle and walks right around the circumference of the smaller circle. After returning to its starting point, the ant walks along a radius to the circumference of the larger circle. The ant now does one circuit of the larger circle and stops. How far has the ant walked?

21. The square of the distance to the horizon in km is approximately equal to 42 times the height of the observer above sea level in metres.
   a. Find a formula for the distance to the horizon in km in terms of the height, h, of the observer in metres.
   b. How much further can a person of 180cm see than a person of 150cm?

22. The sum of a number of consecutive natural numbers is half the number of these numbers times the sum of the smallest and largest of these.
   a. Write a formula for the sum of these numbers.
   b. Use the formula to find the sum of all the natural numbers from 50 to 99.

23. Find a formula for the volume of metal in a uniform cylindrical cup made by removing a cylinder of radius $\frac{7}{8}$R and height $\frac{7}{8}$H from a cylinder of radius R and height H.

24. The sketch shows a running track which consists of two equal semicircular parts of radius R separated by a rectangle of length L.



   a. Find a formula for the circumference of this track.
   b. Find a formula for the area enclosed by this track.
   c. Find a formula for the perimeter of the rectangle.

25. Petrol costs $6.00 per gallon and a car travels 30 miles on one gallon of petrol. Use the approximations 1km = 0.6 miles and 1 gallon = 4.8 litres.
   a. Find the cost of travelling x km.
   b. What would the cost of travelling x km be if the car travels y miles on z gallons?

Changing the Subject of a Formula

The **subject** of a formula is a variable which appears *alone* on one side of the equals sign. The subject is expressed in terms of the other variables. *Changing the subject* of a formula means **rearranging** the formula so that another variable becomes the subject.

Example D: Make x the subject of $y = 2x - 6$.

Solution:

$y = 2x - 6$

$\therefore y + 6 = 2x$ [adding 6 to both sides]

$\therefore 2x = y + 6$

$\therefore x = \frac{y+6}{2}$ [dividing both sides by 2]

Example E: Make T the subject of $\frac{AT+D}{T} = E$.

Solution: $\frac{AT+D}{T} = E$

$\therefore AT + D = ET$ [multiplying both sides by T]

$\therefore ET = AT + D$

$\therefore ET - AT = D$ [collecting terms in T on one side]

$\therefore T(E - A) = D$ [factorising ET – AT]

$\therefore T = \frac{D}{E - A}$ [dividing by E – A]

or $D = 0$ if $E - A = 0$ [substituting E – A = 0 in T(E – A) = D]

Example F: Make M the subject of $\frac{AM^2 - B}{D} = M^2 + E$ when $A \neq D$.

Solution: $\frac{AM^2 - B}{D} = M^2 + E$

$\therefore AM^2 - B = DM^2 + ED$ [multiplying by D]

$\therefore AM^2 = DM^2 + ED + B$ [adding B]

$\therefore AM^2 - DM^2 = ED + B$ [subtracting DM^2]

$\therefore M^2(A - D) = ED + B$ [factorising $AM^2 - DM^2$]

$\therefore M^2 = \frac{ED + B}{A - D}$ [dividing by A – D]

$\therefore M = \pm\sqrt{\frac{ED + B}{A - D}}$ [taking the positive or negative square roots]

(What happens when A = D?)

Using Calculators to Change the Subject

There are a few advanced calculators such as the Hewlett Packard 48G which enable the subjects of many equations to be changed quite simply. Owners of such calculators should consult their manuals.

Exercise 5b: Changing the Subject of Formulae

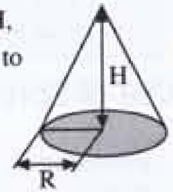
Make M the subject of each of the following formulae:

1. $3M = T$
2. $AM = T$
3. $\frac{M}{P} = T$
4. $\frac{M}{P} = P$
5. $\frac{M}{P} = A + B$
6. $\frac{2M}{P} = 5$
7. $\frac{5M}{T} = T$
8. $M + B = A$
9. $M - B = A$
10. $M - B = B$
11. $M - B - B^2 = B^2 + 2B$
12. $3M - B^2 = 2B^2$
13. $\frac{M}{AB} = BC$
14. $\frac{M}{B} - 3 = C$
15. $\frac{M}{A} = \frac{C}{B}$
16. $M^2 = A$
17. $CM^2 = A + B$
18. $\frac{AM - D}{B} = B$
19. $\frac{AM - DM}{B} = B$
20. $\frac{M}{A} + \frac{M}{B} = C$
21. $\frac{M - A}{M + D} = E$
22. $\sqrt{M} = C$
23. $\frac{\sqrt{M}}{C} = D$
24. $\frac{2A}{3}\sqrt{\frac{M}{G}} = L$
25. $\frac{A}{C}\sqrt{\frac{G}{M}} = X$
26. $\frac{3M^2 - D}{M} = AM$
27. $\frac{A\sqrt{M} - D}{B} = \sqrt{M}$
28. $\frac{A}{\sqrt{M}} + \frac{C}{\sqrt{M}} = E$
29. $y = \frac{AM - 5}{BM - 4}$
30. $y = \frac{3M^3 - E}{4M^3 - F}$

Using Formulae

Formulae need to be used accurately. Using formulae often means **substituting numbers** for some of the variables and solving the equation to find the value of an unknown variable.

Example G: From Example B, the volume of a cone is $\frac{1}{3}\pi R^2 H$, where R is the radius of the base and H is the height. π is taken to be 3.142.



a. Find the volume when R = 2.152cm and H = 4.365cm.
b. Find the radius of the base when the volume is 63.25cm^3 and the height is 5.230cm.

Solution:

a. $V = \frac{1}{3}\pi R^2 H$

$= \frac{1}{3} \times 3.142 \times (2.152)^2 \times 4.365$ [substituting into $V = \frac{1}{3}\pi R^2 H$]

$= 21.17\ cm^3$ [rounding answer to 2 d.p.]

b. $V = \frac{1}{3}\pi R^2 H$

$\therefore 63.25 = \frac{1}{3} \times 3.142 \times R^2 \times 5.230$ [substituting into $V = \frac{1}{3}\pi R^2 H$]

$\therefore R^2 = \frac{3 \times 63.25}{3.142 \times 5.230}$ [solving for R^2]

$R = \sqrt{\frac{3 \times 63.25}{3.142 \times 5.230}}$ [taking the positive square root since R is a length]

$= 3.40$cm (2 d.p.) [rounding off]

Note: *Do not* round off calculations of the types in Example G until the very end of the calculation. Rounding as you progress through a calculation introduces errors at each stage of the calculation, which accumulate over a number of steps producing significant errors in the final answer.

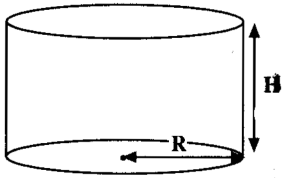
Exercise 5c: Use of Formulae

1. The formula for the area of the walls of a room is $A = 2h(l + w)$ where h is the height of the room, l is the length and w the width.
   a. Find the area if h = 2.3m, l = 5.3m, w = 4.5m.
   b. Find the area if h = 3.41m, l = 4.6m, w = 5.83m.
   c. Find the height if the area if 60.42m^2, the length is 4.86m and the width is 3.92m.
   d. Find the width if the area is 73.4m^2, the height is 4.2m and the length is 6.8m.
   e. Find the length if the area is 165.6m^2, the height is 5.3m and the width is 4.8m.

2. The formula for the volume of a cylinder is $V = \pi R^2 h$, where R is the radius of the cross-section, h is the height and π is taken to be 3.14.
   a. Find the volume if R = 4.6cm, h = 12.3cm.
   b. Find the volume if R = 2.67cm, h = 4.76cm.
   c. Find the height if the volume is $283cm^3$ and the radius is 5.12cm.
   d. Find the radius if the volume is $645cm^3$ and the height is 6.32cm.
   e. Find the radius if the volume is $1367cm^3$ and the height is 15.32cm.

Use of Spreadsheets and Calculators

Example H: The volume of a cylinder is given by the expression $\pi R^2 H$ where R is the base radius and H is the height. (See diagram).



The radius and height are measured to the nearest quarter of a centimetre. Find the maximum volume if the sum of the radius and height has to be 5cm.

	Cylinder Problem		
	A	B	C
	Radius	Height	Volume
4	0.25	4.75	0.933
5	0.5	4.5	3.534
6	0.75	4.25	7.51
7	1	4	12.57
8	1.25	3.75	18.41
9	1.5	3.5	24.74
10	1.75	3.25	31.27
11	2	3	37.7
12	2.25	2.75	43.74
13	2.5	2.5	49.09
14	2.75	2.25	53.46
15	3	2	56.55
16	3.25	1.75	58.07
17	3.5	1.5	57.73
18	3.75	1.25	55.22
19	4	1	50.27
20	4.25	0.75	42.56
21	4.5	0.5	31.81
22	4.75	0.25	17.72
23	5	0	0

Solution: This problem can be tackled by the use of repeated substitution using a non-programmable calculator but can be solved very quickly using a spreadsheet.

This printout shows the results obtained by:

i. using a computer command to generate the values of the radius.

ii. selecting the cells B4 to B23.

iii. typing the formula = 5 – A4 for height into the formula bar.

iv. automatically generating the values below the title **Height**.

v. selecting the cells C4 to C23

vi. typing the formula for volume into the formula bar = 3.1415927 * A4 ^ 2 * B4.

vii. automatically generating the values below the title **Volume**.

The maximum volume is seen to be 58.07 when the radius is 3.25 and the height is 1.75.

Similar results could have been achieved by the use of a programmable calculator although more slowly. A program on a Casio calculator which would enable the solution of this problem is:

$$? \rightarrow R : 5 - R \rightarrow H : 3.1415927 \times R^2 \times H \rightarrow V : V$$

A list of R values is prepared then repeatedly pushing EXE and entering values for R from the list enables the problem to be solved easily. (Owners of other brands of calculators may need to alter the program.)

Problems and Investigations

1. A student is told that she must take two numbers whose sum is nine. They are never allowed to be negative numbers. She is to find the numbers to the nearest one decimal place which give the largest product of one of the numbers with the square of the other. Investigate this problem and find the numbers.

2. A warehouse has two major costs in considering its inventory (the merchandise it contains). These are the ordering cost of $100 per order and the carrying cost which is 0.75% of the maximum value of goods in the warehouse at any time. When the company has decided how many orders it places per year it always orders about the same value of goods per order. During this year it will handle $1 000 000 worth of goods. Investigate how many orders the company should make per year to minimise the cost of its inventory.

3. It is claimed that $T = \left(\frac{B-C}{A} + \frac{C-A}{B} + \frac{A-B}{C}\right)\left(\frac{A}{B-C} + \frac{B}{C-A} + \frac{C}{A-B}\right)$ always has the same value if A + B + C = 0 and A, B, C are all different. Investigate this claim and evaluate T if possible.

6. GRAPHS I

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 4 ALGEBRA

- *to sketch and intepret graphs on whole number grids which represent simple everyday situations*

LEVEL 5 ALGEBRA

- *to sketch and interpret graphs which represent everyday situations*

LEVEL 6 ALGEBRA

- *to form and interpret a graph*

LEVEL 7 ALGEBRA

- *model a variety of situations, using graphs*
- *sketch graphs and investigate the graph of a function, using a calculator and plotting points if necessary*
- *find by inspection, and interpret, maxima, minima, points of inflection, asymptotes and discontinuities for given graphs*

Introduction

A **relation** is any *set of ordered pairs*. In this chapter, relations involving ordered pairs of *numbers* are dealt with. Such relations can be illustrated with a **graph**.

- The **domain** of a relation is the set whose elements appear as the *first* numbers in the ordered pairs of the relation.
- The **range** of a relation is the set whose elements appear as the *second* numbers in the ordered pairs of the relation.
- A **function** is a relation in which each element of the domain occurs as a first number in only *one* ordered pair.
- An **inverse relation** is obtained by *reversing the order* within the ordered pairs of the relation. The inverse of the relation P is commonly written P^{-1}.

Example A: P = {(1, 2), (3, 4), (5, 4)} is a relation.
The domain is the set {1, 3, 5} and range is the set {2, 4}.
The relation is a function because each first number 1, 3, and 5 in the three pairs occurs in only one ordered pair of the relation.
The inverse relation is P^{-1} = {(2, 1), (4, 3), (4, 5)}.

Use of Graphical Calculators

Most graphical calculators have a mode which enables the points to be plotted. For a student using a Casio graphical calculator they would have to put their calculator into the graph mode REC/PLT which enables the points to be plotted using a pair of x, y co-ordinate axes. The student then has to select a suitable domain and range using the RANGE key. Points are then simply plotted using the PLOT key.

Exercise 6a

1. For the relation T = {(1, 3), (2, 5), (3, 7), (4, 11), (5, 15)}.
   a. Write down the domain.
   b. Write down the range.
   c. Draw a neat graph. [Note: As there are only five points on this graph the points should not be connected.]
   d. Explain why the relation is a function.
   e. Draw a graph of the inverse relation T^{-1}.
   f. Is T^{-1} a function? Explain your answer.

2. For the relation A = {(2, 3), (3, 3), (4, 5), (5, 6), (6, 7)}.
   a. Write down the domain.
   b. Write down the range.
   c. Draw a neat graph.
   d. Explain why A is a function but A^{-1}, its inverse relation, is not.

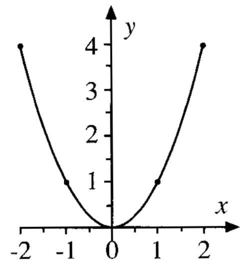
Drawing Graphs

The following example is in **set builder notation** and illustrates the general method for drawing graphs.

Example B: Find the domain and range and draw the graph of the function:
$\{(x, y): y = x^2, -2 \le x \le 2.\}$

Solution: The domain is the set $-2 \le x \le 2$.
The ordered pair corresponding to any x value is (x, y) where $y = x^2$.
The graph is drawn by setting up a **table** of sensibly chosen x values (which are values from the domain) and calculating the corresponding y values (which are values in the range).

x	-2	-1	0	1	2
y	4	1	0	1	4



The range shown by the graph is $0 \le y \le 4$.

The points in the table are **plotted** with the domain on the horizontal axis. Where sensible, the points are joined by a smooth curve.

Note:

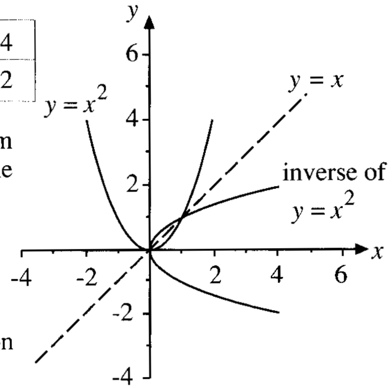
a. The relation is a function because there is only one point on the graph which is directly above any point in the domain. Generally a graph will represent a function if any vertical line cuts the graph at no more than one point. [This is known as the **vertical line test.**]

b. The range is the set of all values on the vertical axis through which a horizontal line can be drawn which will touch or cut the graph.

Example C: Draw the inverse relation of the function $y = x^2$, $-2 \le x \le 2$.

Solution: Using the table from Example B, the values are reversed:

x	4	1	0	1	4
y	-2	-1	0	1	2



Plotting these co-ordinates and joining them gives the following graph. The graphs of the relations $y = x$, $y = x^2$ and the inverse of $y = x^2$ are shown for comparison.

Note:

a. Each graph is the *reflection* of the other in the line $y = x$.

b. The inverse relation is *not* a function because any vertical line drawn through the graph will cut it twice (except when $x = 0$).

c. The domain of the inverse relation is $0 \le x \le 4$ (which is the range of $y = x^2$) and the range is $-2 \le y \le 2$ (which is the domain of $y = x^2$).

When no domain is specified it is assumed that the domain is the *largest set of real numbers* possible for which the expression is defined.

The point(s) where a graph cuts the x axis is the ***x* intercept**(s). Similarly, the point(s) where a graph cuts the y axis is the ***y* intercept**(s).

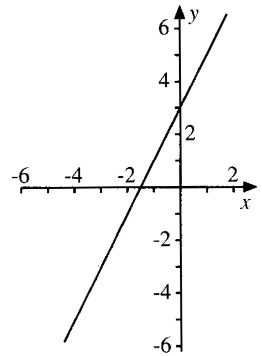
Example D: Draw the graph of the function $f: x \rightarrow 2x + 3$.

Solution: This function is described in **mapping notation**. It is read 'f takes x to $2x + 3$' and means exactly the same as $y = 2x + 3$.

The domain is all **real numbers** (i.e. the set R) because any real number can be doubled and have 3 added to it.

A suitable table with some points plotted on a graph is shown:

x	-1 000	-10	-3	-1	0	1	3	10	1 000
y	-1 997	-17	-3	1	3	5	9	23	2 003



Note:

a. The graph is a straight line.

b. The range is R.

c. Often values in the table cannot be plotted on the graph. Such values are still valuable because they show the behaviour of the function as x gets particularly large or small.

d. The graph has an x intercept of $-1\frac{1}{2}$ because the graph crosses the x axis at the point $(-1\frac{1}{2}, 0)$.

e. The graph has a y intercept when $y = 3$ because the graph crosses the y axis at the point (0, 3). The x and y intercepts are useful when sketching a graph.

Use of Graphical Calculators

In problems where any domain is given, graphical calculators make the drawing of graphs particularly simple. Using a Casio graphical, the RANGE key is used to input the domain required into the calculator. By use of the GRAPH key the equation of the graph is input, the EXE key is pressed, and the graph is drawn. By use of the TRACE key the coordinates of any point on the graph can be found, including x and y intercepts. By drawing the graph on such a calculator then using the TRACE key, the coordinates of points on the graph can be transferred onto paper and the graph drawn.

Exercise 6b

1. a. Draw the graph of $y = 9 - x^2$, $-4 \le x \le 4$.
   b. What are the x intercepts?
   c. What is the y intercept?

2. a. Draw the graph of $y = (x + 1)(x - 1)(x - 2)$, $-2 \le x \le 3$.
   b. What are the x intercepts?
   c. What is the y intercept?

Features of Graphs

- A function is **increasing** if the values in its range increase as the domain values increase.
- A function is **decreasing** if the values in its range decrease as the domain values increase.
- A function has a **maximum turning point** on its graph when it changes from being an increasing function to being a decreasing function.
- A function has a **minimum turning point** on its graph when it changes from being a decreasing function to being an increasing function. The **nature of a turning point** is determined by whether the turning point is a maximum or minimum.
- A function has **point symmetry** if the graph of the function is able to rotate through 180° about a point onto itself.

Example E: The function f is given by f: $x \rightarrow -x^3 + 9x$.

a. Copy and complete the table of values for f:

x	-4	-3	-2	-1	0	1	2	3	4
y	28	0	-10						

b. Draw the graph of function f.
c. i. Write down the maximum and minimum turning points of f.
   ii. Use inequality signs to give the values of x for which f is increasing.

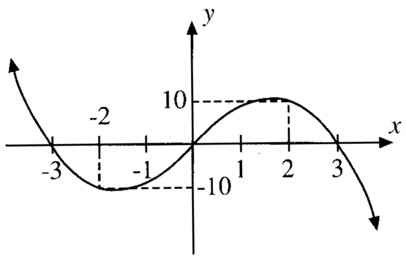
Solution:

a.

x	-4	-3	-2	-1	0	1	2	3	4
y	28	0	-10	-8	0	8	10	0	-28

Note: The arrows are optional. (They are used to show that the graph continues indefinitely.)

b. See graph.



c. i. The function has a maximum turning point at (1.7, 10.4) and a minimum turning point at (-1.7, -10.4). (These would be easy for the reader to see if a more accurate graph is plotted using more points than are given in the table.)
   ii. The function is increasing for $-1.7 < x < 1.7$ and decreasing for $x < -1.7$ and for $x > 1.7$.

Note:

a. The graph has x intercepts when $x = -3$, 0 and 3.
b. The graph has a y intercept when $y = 0$.
c. $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$. This is read 'f(x) tends to negative infinity as x tends to infinity' and means that as x gets larger, y gets larger without limit in a negative sense.
d. $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$. This is read as 'f(x) tends to infinity as x tends to negative infinity' and means that y gets larger without limit as x gets larger in a negative sense.
e. The graph has point symmetry about the point (0, 0) because if the graph is rotated 180° about (0, 0) it will rotate onto itself.

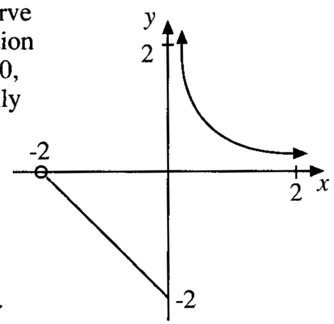
A **continuous** graph consists of one unbroken line, curve, or combination of both. All the graphs in the previous examples are continuous.

A point of **discontinuity** is a point in the domain where there is a 'break'. The graph of such a function is described as being **discontinuous,** which means that it is broken into at least two separate sections which do not touch each other.

Example F: The function $y = f(x)$ is represented by the graph below.

Note:

a. The 'hole' (represented by o) at (-2, 0) shows that the point (-2, 0) is *not* on the graph.
b. The arrow (optional) on the top of the curve in the right quadrant shows that the function values get ever larger as x gets closer to 0, for the positive values of x. This is usually written $f(x) \rightarrow \infty$, as $x \rightarrow 0^+$.
c. As $x \rightarrow \infty$, $f(x) \rightarrow 0$. This is read 'as x tends to infinity y or f(x) tends to 0'. This means that y gets closer to 0 as x gets larger and larger. This is shown on the graph by the arrow on the bottom of the curve in the right quadrant.



d. The graph is discontinuous.
e. The domain is $x > -2$, or in set builder form $\{x : x > -2\}$.
f. The range is written $\{y : -2 \le y < 0 \text{ or } y > 0\}$ or $\{y : y \ge -2,\ y \ne 0\}$.

Use of Graphical Calculators

Using the RANGE and GRAPH keys, graphs of all functions can be easily drawn. The following information will then be easily found with the help of the TRACE key:

- co-ordinates of turning points
- sets on which functions are increasing and decreasing
- points about which there is point symmetry
- points of discontinuity

Exercise 6c

1. a. Draw a neat graph of the function: $y = 4 - x^2$, $-2 \le x \le 3$.
   b. What are the x intercepts?
   c. What is the y intercept?
   d. What is the domain?
   e. What is the range?
   f. For what values is the function decreasing?
   g. Draw the graph of the inverse relation on the same axes.

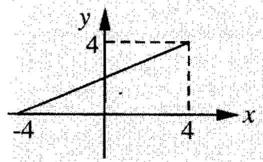
2. a. Draw a neat graph of the function: $y = (x-1)^2 + 3$, $-1 \le x \le 3$.
   b. What are the co-ordinates of the turning point of this function? Is it a maximum or minimum?
   c. For what set of values is the function increasing?
   d. What are the co-ordinates of the y intercept?
   e. Is the graph symmetrical in any way?

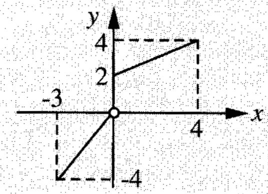
3. a. Draw a neat graph of the function: $y = x(x^2 - 1)$, $-2 \le x \le 2$.
   b. Write down the approximate co-ordinates of the turning points.
   c. Describe the nature of any turning points.
   d. Describe any symmetry of this graph.
   e. What are the x intercepts?
   f. What is the y intercept?
   g. What is the domain and range of this function?

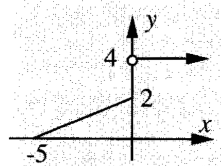
4. a. Draw a neat graph of $(1-x)x^2$, $-2 \le x \le 2$.
   b. What are the x and y intercepts?
   c. Write down the co-ordinates of any turning points indicating their nature.
   d. Indicate for what values the function is decreasing.
   e. Does this graph have any symmetry?
   f. What is the domain and range of this function?

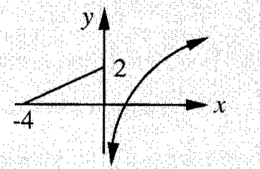
5. a. Draw a neat graph of $f : x \to \frac{x^3}{4}$.
   b. Does this graph have any symmetry?
   c. What does the graph do as $x \to \infty$ and as $x \to -\infty$?
   d. By drawing the graph of $y = x$ or otherwise solve the equation $\frac{x^3}{4} = x$.
   e. Write down the domain and range of f.

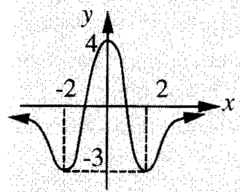
For the functions sketched below write down the domain and range, x and y intercepts, turning points, symmetries, behaviour as x tends to $\pm\infty$ where applicable, any points where the function tends to $+\infty$ or $-\infty$, or any discontinuities.

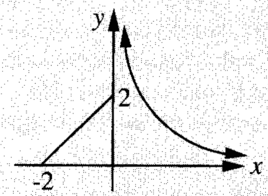
6.


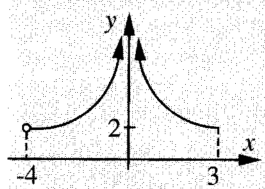
7.


8.


9.


10.


11.


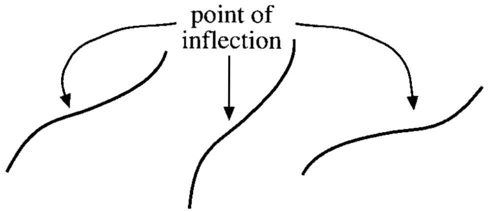
12.


Other Features of Graphs

a. Points of Inflection

A **point of inflection** is a point of a graph where the curve is steepest or shallowest in its neighbourhood yet does not change from being either increasing or decreasing. The graph appears to 'wriggle' at such a point.

Example G:

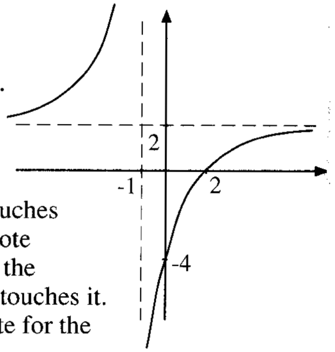


b. Asymptotes

An **asymptote** is a line to which a graph gets ever closer but never actually touches.

Example H: Consider the function $y = \frac{2x - 4}{x + 1}$.

The graph is shown:



As x gets closer and closer to -1 the graph gets closer and closer to the line $x = -1$ but never touches it. The line $x = -1$ is called the vertical asymptote for the function. As $x \to \pm\infty$ (gets ever larger) the graph gets closer and closer to $y = 2$ but never touches it. The line $y = 2$ is called the horizontal asymptote for the function.

Exercise 6d

1. For the function $g(x) = \frac{1}{x + 1}$.
   a. Draw a neat graph.
   b. Are there any turning points?
   c. What are the x and y intercepts?
   d. Describe the behaviour of $g(x)$ as x gets close to -1 by completing the following sentences.
   $g(x)$.......... as $x \to -1^-$ and $g(x)$.......... as $x \to -1^+$
   e. Does the graph have any symmetry?
   f. What are the domain and the range?
   g. What are the horizontal and vertical asymptotes of the graph?
   h. Are there any points of inflection? Explain.

2. For the graph $y = 1 + \frac{1}{x + 2}$.
   a. Draw a neat graph.
   b. What are the domain and range?
   c. What value does y tend to as x tends to ∞ and to $-\infty$?
   d. For what value in the set of real numbers is y not defined?
   e. Describe the behaviour of the graph near the point mentioned in (d).
   f. What are the vertical and horizontal asymptotes?
   g. Are there any points of inflection? Explain.

3. a. For the function $g(x) = \frac{4}{1 + x^2}$ fill in this table.

x	-100	-5	-3	-1	0	1	3	5	100
y									

   b. Using your table draw a neat graph of this function.
   c. What are the domain and range of this function?
   d. Describe its behaviour as $x \to \infty$ and as $x \to -\infty$.
   e. For what values is it increasing?
   f. Are there any asymptotes? If so, give their equations.
   g. How many points of inflection are there?

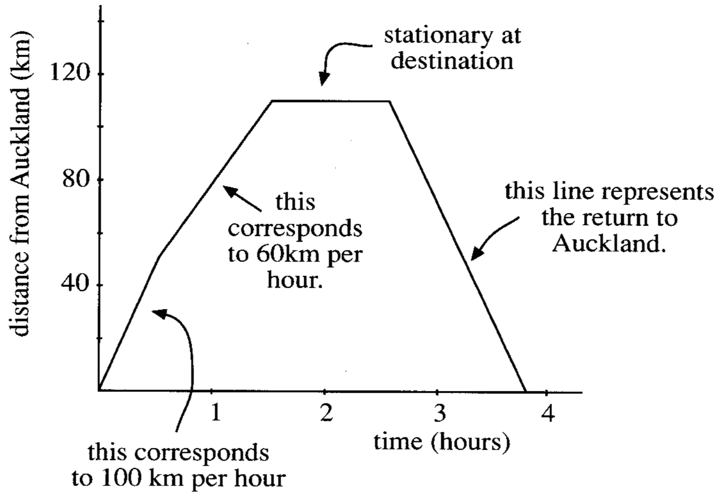
Applications of Graphs

Graphs provide a very useful way of transmitting important information.

Example I: A traveller set out from Auckland to Hamilton at a steady rate of 100km per hour. After half an hour he reduced his speed to 60km per hour which he maintained for the next hour, at which time he reached his destination. He spent an hour there then returned to Auckland, travelling at 90km per hour. Draw a graph showing distance against time.

Solution: As with the preceding work on graphs it is a good idea to set up a table then plot the resulting points.

Time (hrs)	0	0.25	0.5	0.75	1	1.25	1.5	2.0	2.5	3.0	3.5
Distance from Auckland (km)	0	25	50	65	80	95	110	110	110	65	20



The graph consists of 4 distinct parts as shown:

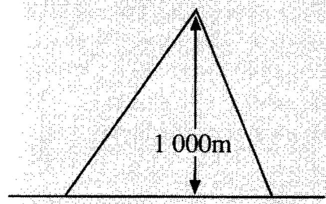
- The first part for times between 0 and 0.5 hours represents that part of the journey where the car was moving away from Auckland at 100km per hour.
- The second for times between 0.5 and 1.5 represents that part of the journey where the car was moving away from Auckland at 60km per hour. Notice that the graph is increasing in these two parts as the distance from Auckland is increasing.
- Between 1.5 and 2.5 the car is at rest and the distance from Auckland is constant.
- From 2.5 till journey's end the distance from Auckland is decreasing and so is the graph.
- The time when the traveller returns to Auckland is found where the graph cuts the horizontal axis. It is about 3.7 hours after leaving Auckland.

Use of Graphical Calculators

The use of graphical calculators greatly simplifies this sort of problem. Use of range, graph, plot, line and trace keys greatly assist in the drawing and interpretation of graphs such as those in example I.

Exercise 6e

1.



A student climbs up the side of the hill shown in the diagram then comes down the other side.

During the climb up the hill the student rises by 10m per minute. During the climb down the hill the student descends by 40m per minute.

a. Copy and fill in the table of values.

Time (minutes)	0	25	50	75	100	115	125
Height above base of hill (m)	0						

b. Draw a neat graph of height against time.
c. How high was the student above the base of the hill at the following times:
 i. 30 minutes. ii. 80 minutes. iii. 112 minutes.
d. At what time was the student at a height of 800m above the base of the hill?
e. During what times was the student:
 i. rising? ii. descending?
f. Write down the domain and range of the function which relates time to height.

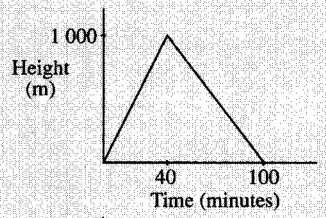
2. Another student climbs the same hill. During the climb up the hill this student rises 20m per minute for the first 25 minutes then 10m per minute for the rest of the climb. The student also descends by 40m per minute after reaching the top.
a. How long does it take this student to reach the top of the hill?
b. Set up an appropriate table of values for the climb.

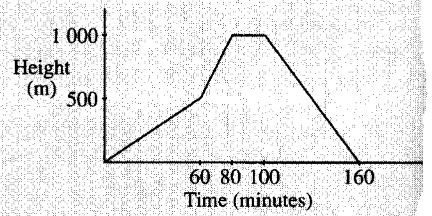
Time (minutes)									
Height above base of hill (m)									

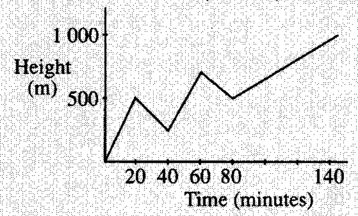
c. Draw a neat graph of height against time.
d. How high was the student above the base of the hill at the following times:
 i. 30 minutes. ii. 80 minutes. iii. 112 minutes.
e. At what times was the student at a height of 800m above the base of the hill?
f. During what times was the student:
 i. rising? ii. descending?
g. Write down the domain and range of the function which relates time to height.

3. A third student climbs the same hill. This student rises by 25m per minute for the first 20 minutes then 15m per minute for the next 25 minutes then 10m per minute for the rest of the climb. This student also descends by 40m per minute after reaching the top of the hill.
   a. Draw a neat graph of height against time.
   b. Draw graphs which would show the following changes in this student's climb:
      i. The student climbed up and down at the same rates but takes a ten minute rest before climbing down.
      ii. The student climbed up the hill at the same rates but then came down (without taking a rest) at 80m per minute.

4. The following graphs represent the climbs of different students up the hill referred to in questions 1, 2 and 3.

a.


b.


c.


For each graph describe the climb it represents.

Problems and Investigations

1. A bag contains pieces of wire 40cm long which are bent into rectangles.
   a. If one of the sides of a rectangle is 5cm long:
      i. How long is the other side?
      ii. What is the area of the rectangle?
   b. Fill in the table of values.

Length of side (cm)	1	2	4	6	8	10	12	14	16	18
Area of rectangle (cm^2)										

   c. Draw a neat graph of area against side length.
   d. What are the domain and range of the function which relates side length to area?
   e. Find the maximum area of the function which relates side length to area.

2. A tank contains 10 litres of fluid. During the first minute one half of the fluid in the tank is drained off. During the next minute one third of what is left is drained off. During the next minute one quarter of what remains is drained off. This process is continued for 15 minutes.
   a. Draw a graph of the volume of fluid in the tank against time.
   b. How many minutes pass by before there is less than one litre of fluid in the tank?

7. SIMULTANEOUS EQUATIONS

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 6 ALGEBRA

- *to form and solve simultaneous equations*

LEVEL 7 ALGEBRA

- *to choose suitable strategies for finding solutions to equations (including pairs of simultaneous equations, one of which may be non-linear)*

Introduction

Simultaneous equations involve two or more unknowns. Simultaneous equations with *two* unknowns are discussed in this chapter.

Linear Equations

One procedure for solving a pair of linear equations involves **elimination**. This means that the equation is solved by adding together (or subtracting) the equations to eliminate an unknown. Sometimes one (or both) of the equations must be multiplied by a constant before the equations are added or subtracted so that the unknown has the *same coefficient* in both equations.

Example A: Solve the equations simultaneously: $2x + 3y = 11$ and $x + 4y = 13$

Solution: The equations are numbered for easy identification.

$$2x + 3y = 11 \quad \text{.... (1)}$$
$$x + 4y = 13 \quad \text{.... (2)}$$

To eliminate y, (1) is multiplied by 4 and (2) is multiplied by 3. When the resulting equations are subtracted, the $12y$ in each equation cancels.

$$\begin{array}{lll} (1) \times 4 & 8x + 12y = 44 & \text{.....(3)} \\ (2) \times 3 & \underline{3x + 12y = 39} & \text{.....(4)} \\ & 5x \quad\quad = 5 & \text{[subtracting (4) from (3)]} \\ & \therefore\ x = 1 & \end{array}$$

The value of y is found by substituting the value $x = 1$ into either one of the original equations (1) or (2):

$$\begin{array}{ll} 2x + 3y = 11 & \text{.... (1)} \\ \therefore\ 2 + 3y = 11 & \text{[substituting } x = 1 \text{ into (1)]} \\ \therefore\ 3y = 9 & \text{[subtracting 2]} \\ \therefore\ y = 3 & \end{array}$$

The solution for the equations $2x + 3y = 11$ and $x + 4y = 13$ is $x = 1$ and $y = 3$.

Note: An alternative method of solution would be the elimination of x to find y.

In some problems the denominators are removed and like terms are gathered before any elimination occurs. Two eliminations are then done, one for each of the unknowns.

Example B: Solve $\frac{3A - 2B + 7}{5} = \frac{4A + 3B - 11}{2}$ and $\frac{1}{2}(2A + 5B) = 8$

Solution: $\frac{3A - 2B + 7}{5} = \frac{4A + 3B - 11}{2}$

$$\begin{array}{lll} \therefore\ 6A - 4B + 14 = 20A + 15B - 55 & & \text{[multiplying by 10]} \\ \therefore\ 14A + 19B = 69 & \text{..... (1)} & \text{[gathering like terms]} \\ \text{and } \frac{1}{2}(2A + 5B) = 8 & & \\ \therefore\ 2A + 5B = 16 & \text{..... (2)} & \text{[multiplying by 2]} \end{array}$$

A is now eliminated:

$$\begin{array}{llll} (1) & 14A + 19B = 69 & \text{..... (1)} & \\ (2) \times 7 & 14A + 35B = 112 & \text{..... (3)} & \\ & 16B = 43 & & \text{[subtracting (1) from (3)]} \\ & \therefore\ B = 2\frac{11}{16} & & \end{array}$$

The problem can be finished by substituting $B = 2\frac{11}{16}$ into (1) or (2) to find A. A simpler solution involves eliminating B in order to find A.

$$\begin{array}{llll} (1) \times 5 & 70A + 95B = 345 & \text{..... (4)} & \\ (2) \times 19 & \underline{38A + 95B = 304} & \text{..... (5)} & \\ & 32A \quad\quad = 41 & & \text{[subtracting (4) from (5)]} \\ & \therefore\ A = \frac{41}{32} & & \\ & = 1\frac{9}{32} & & \end{array}$$

$\therefore$ The solution for the simultaneous equations $\frac{3A - 2B + 7}{5} = \frac{4A + 3B - 11}{2}$ and $\frac{1}{2}(2A + 5B) = 8$ is $A = 1\frac{9}{32}$ and $B = 2\frac{11}{16}$.

Sometimes a pair of simultaneous equations does not have a solution.

Example C: $2x + 3y = 2$ .... (1)

$4x + 6y = 5$ .... (2)

Attempted Solution:

(1) × 2 $\quad 4x + 6y = 4$ .... (3) [eliminating x]

(2) $\quad 4x + 6y = 5$ .... (2)

$0 = -1$ [subtracting (2) from (3)]

Since $0 \neq -1$, there is no solution. The result corresponds to two straight line graphs which are parallel and do not intersect.

Sometimes there is an infinite set of solutions.

Example D: Solve $3x - 2y = 4$ .... (1)

$9x - 6y = 12$ .... (2)

Solution: If $3x - 2y = 4$ is multiplied by 3 we get $9x - 6y = 12$. This means that the two equations represent the same set of points and thus there will be an infinite set of solutions to this pair of simultaneous equations.

Solution of Simultaneous Linear Equations by the Method of Substitution

An alternative method to solving simultaneous linear equations is that of **substitution**. The following example illustrates its use.

Example E: Solve $2y - 5x = 3$

$x + y = 5$

Solution: i. Make y the subject of the first equation.

$\therefore y = \frac{3 + 5x}{2}$ [you could use the second equation and you could start with x instead]

ii. Substitute for y in the second equation.

$\therefore x + \frac{3 + 5x}{2} = 5$

iii. Now solve the equation in (ii).

$\frac{2x + 3 + 5x}{2} = 5$ [adding x and $\frac{3 + 5x}{2}$]

$\frac{7x + 3}{2} = 5$ [simplify]

$\therefore 7x + 3 = 10$ [cross multiply]

$\therefore x = 1$

iv. Substitute this value of x in the equation in (i).

$y = \frac{3 + 5}{2} = 4$

Solution of Simultaneous Linear Equations using Calculators

A number of calculators give the solution of linear simultaneous equations.

Example F: Solve $2x + 3y = 7 + 2y$

$3x - 2y = 9 - x + y$

Using a. Casio fx -7700GB

b. HP48G

Solution:

a. It is necessary to write the equation in the form $\bigcirc x + \square y = \Delta$, where $\bigcirc$, $\square$ and Δ are numbers. $2x + 3y = 7 + 2y$ becomes $2x + y = 7$ (subtracting $2y$ from both sides of '=').

Similarly $3x - 2y = 9 - x + y$ becomes $4x - 3y = 9$. The coefficients of x, y then form a rectangular array of numbers $\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$ called a 2 x 2 matrix. Similarly the numbers on the right side of the '=' sign form another rectangular array $\begin{pmatrix} 7 \\ 9 \end{pmatrix}$ called a 2 x 1 matrix.

The next steps are simple.

i. $\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$ is entered into matrix A in the calculator.

ii. $\begin{pmatrix} 7 \\ 9 \end{pmatrix}$ is entered into matrix B.

iii. The inverse of $\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$ is found by pressing a certain key.

iv. The inverse 'multiplies' B and gives the answer $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ which is the solution $x = 3$, $y = 1$.

b. *Solution using an HP48G.* Solving using a HP48G also involves entering $\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}$ as A and $\begin{pmatrix} 7 \\ 9 \end{pmatrix}$ as B. Once done a solve command is carried out and the solution $x = 3$, $y = 1$ is obtained.

Exercise 7a: Two linear equations

Solve each of the following pairs of simultaneous linear equations:

1. $x + 3y = 9$ $4x + y = 14$	2. $5x + y = 3$ $2x + y = 3$	3. $3U + 5V = 28$ $2U - 3V = -13$
4. $T - 6R = 14$ $T + 5R = -8$	5. $8A + 7B = 42$ $2B - 2A = -3$	6. $3y - 5x = 1$ $4y + x = 7$
7. $2(2y + x) = 5x + 15$ $3y - x = 4x + y - 3$	8. $2(x + y) = 5 + x$ $3(x - 2y) = 55 - 2y$	
9. $3y + x - 3 = 3x + 2y - 2$ $2(x - y) = 3(x - y) - 1$	10. $\frac{y - x}{2} = \frac{y + x + 1}{3}$ $2x - \frac{y}{3} = 2$	11. $\frac{y}{3} + \frac{x}{5} = 1\frac{1}{2}$ $\frac{3y}{4} - \frac{x}{2} = 1$
12. $\frac{1}{8}(x + y) = \frac{1}{3}$ $\frac{1}{4}y - \frac{3}{5}x = 1.8$	13. $0.23x + 1.32y = 4.56$ $0.07x - 2.51y = 3.17$	14. $\frac{x - 3}{y} = 2$ $\frac{x + 5}{3y} = 2$
15. $3x + 5y = 2$ $4.5x + 7.5y = 3$	16. $\frac{1}{x} - y = 5$ $\frac{1}{x} + y = 6$	17. $x^2 - y^2 = 3$ $x^2 + 2y^2 = 6$
18. $\frac{2}{x} - \frac{3}{y} = 1$ $\frac{3}{x} + \frac{1}{y} = 7$	19. $\sqrt{x} + 3y^2 = 11$ $3\sqrt{x} - y^2 = 3$	20. $\frac{4}{\sqrt{x}} - \sqrt{y} = 10$ $\frac{3}{\sqrt{x}} + 2\sqrt{y} = 13$

Word Problems

Example G: Jo and Sally together earn \$132 an hour. Jo earns 20% more than Sally. What are their hourly earnings?

Solution: Let Jo earn J dollars and Sally earn S dollars per hour.

This gives: $S + J = 132$ .... (1) and $J = 1.2S$ .... (2)

$S + 1.2S = 132$ [substituting for J in (1)]

$\therefore\ 2.2S = 132$

$\therefore\ S = \frac{132}{2.2}$

$= 60$ ie Sally earns \$60 per hour

$J + 60 = 132$ [substituting in (1)]

$\therefore\ J = 72$ ie Jo earns \$72 per hour

Exercise 7b: Word Problems

Write simultaneous equations for each of the following problems and then solve the equations.

1. The sum of two numbers is 138 and the difference is 12. Find the two numbers.

2. A pupil spends one third less time on her Maths homework than she does on her English homework. The total time spent on both subjects is $2\frac{1}{2}$ hours. How much time does she spend on each subject?

3. Product A is twice as expensive as Product B. A company buys three times as much of B as it does of A and spends \$52 in purchasing an order of these two products. This order includes 20 kg of Product A. What is the cost per kg of each product?

4. Even if Peter was to increase his earnings by 40% he would still be earning \$140 a week less than Michael. A company employs two people earning Michael's wages and five earning Peter's wages. It pays out \$4 960 in wages per week. How much do Peter and Michael get each week?

5. At the moment, Boris Holdings owns eleven more properties than A.C. Realty. Last year Boris owned twice as many as they do now, and A.C. three times as many. At that time Boris owned ten more than A.C. How many properties do Boris Holdings and A.C. Realty currently own?

6. Each time Mary goes for a run she runs two km more than Sue. Mary runs seven days a week and Sue runs six days a week. Mary runs 22 km a week more than Sue. How far does each woman run per day?

7. A room contains 96 people. A man leaves and is replaced by a woman, leaving three times as many women as men in the room. How many men and how many women were there originally?

8. $y = Ax^2 + Bx + 5$ is a parabola which goes through the points (2, 5) and (5, 2). Find the values of A and B.

9. $y = \frac{x + A}{x + B}$ is a hyperbola which goes through (2, 5) and (5, 2). Find the values of A and B.

10. $y = x^3 + Ax^2 - 7x + B$ is a cubic polynomial whose graph goes through the points (1, 0) and (-3, 0).
    a. Find A and B.
    b. $x^3 + Ax^2 - 7x + B$ can be written in the form $(x - 1)(x + 3)(x - c)$. Find c.

Solving a Linear and Quadratic Equation

The method used to solve a linear equation and a quadratic equation simultaneously is to **substitute** one of the variables of the linear equation into the quadratic equation. If necessary, one of the variables will need to be made the subject of the linear equation.

Example H: Solve $y = x + 1$ and $y^2 - 3x = 13$.

Solution: y is the subject of the linear equation.

$\therefore\ (x+1)^2 - 3x = 13$ [substituting $(x + 1)$ for y in the quadratic]

$\therefore\ x^2 + 2x + 1 - 3x = 13$ [removing brackets]

$\therefore\ x^2 - x - 12 = 0$ [simplifying]

$\therefore\ (x-4)(x+3) = 0$ [factorising]

$\therefore$ either $(x - 4) = 0$ or $(x + 3) = 0$

$\therefore\ x = 4$ or $x = -3$

when $x = -3$, $y = -3 + 1 = -2$ [substituting into $y = x + 1$]

when $x = 4$, $y = 4 + 1 = 5$ [substituting into $y = x + 1$]

Example I: Solve $2x - 3y = 6$ ....(1) and $x^2 + y^2 = 9$ ....(2).

Solution:

$2x - 3y = 6$ ....(1)

$\therefore\ 2x = 3y + 6$ [adding $3y$]

$\therefore\ x = \frac{3y+6}{2}$ [dividing by 2]

$\therefore\ \left(\frac{3y+6}{2}\right)^2 + y^2 = 9$ [substituting for x in (2)]

$\therefore\ \frac{9y^2+36y+36}{4} + y^2 = 9$ [expanding]

$\therefore\ 9y^2 + 36y + 36 + 4y^2 = 36$ [multiplying by 4]

$\therefore\ 13y^2 + 36y = 0$ [simplifying]

$\therefore\ y(13y + 36) = 0$ [factorising]

$\therefore\ y = 0$ or $13y + 36 = 0$

$\therefore\ y = 0$ or $y = \frac{-36}{13}$

When $y = 0$, $x = 3$ and when $y = \frac{-36}{13}$, $x = \frac{-15}{13}$. [substituting into $x = \frac{3y+6}{2}$]

Exercise 7c: One Linear Equation and One Quadratic Equation

Solve each of the following pairs of simultaneous equations:

1. $y = x - 1$, $x^2 + y^2 = 1$
2. $y = 2x + 1$, $x^2 + y^2 = 1$
3. $y = x + 2$, $y^2 + 2x^2 = 4$

4. $y = 2x - 3$, $x^2 + y^2 = 9$
5. $2y - 3x = 6$, $x^2 + y^2 = 9$
6. $y = x + 1$, $x^2 + y^2 = 5$
7. $y = x - 3$, $x^2 + y^2 = 29$
8. $x - 5y = 12$, $y^2 = 2x$
9. $4x + 3y = 25$, $xy = 12$
10. $2x + y = 7$, $xy = 6$
11. $x^2 + y^2 = 13$, $y = x + 1$
12. $x^2 + y^2 = 5$, $y + x = 3$
13. $xy = 8$, $x - y = 2$
14. $y^2 = 3x$, $3x - 2y = 8$
15. $y = \frac{1}{x+1}$, $2y + 3x = 4$

Problems and Investigations

1. Find a pair of simultaneous equations which have the following as solution.
   a. $x = 3, y = 2$
   b. $x = 5, y = -3$

2. Find the number of novels in a small library if:
   a. The number exceeds 15 as does the number of other books.
   b. The number of novels is prime.
   c. The total number of books in the library is 40.
   d. The number of novels is less than the number of other books.
   e. If the number of novels was increased by 1 and so was the number of other books then the number of novels and the number of other books would share a common factor bigger than 2.

3. Find the integers x, a satisfying this equation.

   $$x^a + x^{a+1} + x^{a+2} = 55\,728$$

4. Find the integers x, y satisfying this equation.

   $$3x + 2y = 81$$
   $$\frac{1}{x} + \frac{1}{y} = \frac{32}{255}$$

5. Find the whole numbers x, y, z which satisfy these equations.

   $$2x + 4y + z = 83$$
   $$\frac{z}{x} + \frac{1}{y} = \frac{12}{5}$$

8. SOLUTION OF POLYNOMIAL EQUATIONS

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 6 ALGEBRA

- *to form and solve simple quadratic equations*

LEVEL 7 ALGEBRA

- *to choose suitable strategies for finding solutions to equations and interpret the results*

  (Suggested learning experience: students should be investigating real and simulated situations including the solution of polynomial equations and the nature of their roots)

Introduction

A **polynomial** is an expression made up of a sum of **terms**. Each term is a multiple of a power of the same variable. The powers are *whole numbers*. The **degree** of a polynomial is the *highest* power of the variable present in any of its terms.

Example A: $P(x) = 2x^4 - 3x^3 + 2x^2 + 5x - 7$ is a polynomial function containing five terms: $2x^4$, $-3x^3$, $2x^2$, $5x$ and -7. The polynomial has degree 4 because the highest power of x is 4 which occurs in the term $2x^4$. ($P(x) = (x + 5)^2$ is a polynomial because it expands to $x^2 + 10x + 25$.)

This chapter deals with polynomials where the power of the variable is greater than 1. Two particular cases occur when the polynomial has degree 2 (described as **quadratic**) and degree 3 (described as **cubic**).

Exercise 8a: Polynomials

1. Which of the following are polynomials?

a.	$2x^2 + 3x - 1$	b.	$4x^3 - 5x + 2$	c.	$x^2 - 3$
d.	$\dfrac{1}{x^2+1}$	e.	$x^2 + x^{-1}$	f.	$\frac{1}{2}x^2 + 3x - 2\frac{1}{3}$
g.	$\sqrt{x}$	h.	7	i.	$(x + 3)^2$
j.	$4(x + 3)^2$				

2. What is the degree of each of the following polynomials?

a.	$2x^3 - x + 3$	b.	$x^5 - 2x^4 + x^3 + 2x^2 - 4x + 3$		
c.	$\frac{1}{3}x^6 - 4x + 2$	d.	$12x^4 - 3x^2 + x$	e.	$3x + 4$
f.	$(x + 3)(x + 5)$	g.	$x(x + 3)^2$	h.	$(2x^2 - 1)^2$
i.	$4 - x^2$	j.	5		

3. Which of the following polynomials are quadratics? Which are cubics?

a.	$2x^2 + 3x + 5$	b.	$8x^3 - 4x + 5$	c.	$2 - 3x^2$
d.	$2x^4 + x^2 + 1$	e.	$(x + 1)^2$	f.	$(x + 3)(2x + 1)$
g.	$x^2(x - 1)$	h.	$x^6 - 5x^2 + 7$		

Calculations with Polynomials

Various calculations are possible with polynomials. These include substitutions and equation solving as shown in the following examples.

Example B: If $h(x) = 4x^3 - 2x^2 + 5x - 3$, h(-2) is the value of the polynomial when $x = -2$. It can be found by substituting $x = -2$ into the formula for $h(x)$.

$$h(-2) = 4(-2)^3 - 2(-2)^2 + 5(-2) - 3 \quad \text{[substituting]}$$
$$\therefore h(-2) = -32 - 8 + {}^{-}10 - 3 \quad \text{[simplifying]}$$
$$= -53$$

$h(\frac{1}{2})$ is the value of the polynomial when $x = \frac{1}{2}$. $h(\frac{1}{2})$ can be found by substituting $x = \frac{1}{2}$ into the formula for $h(x)$.

$$h(\tfrac{1}{2}) = 4(\tfrac{1}{2})^3 - 2(\tfrac{1}{2})^2 + 5(\tfrac{1}{2}) - 3 \quad \text{[substituting]}$$
$$= \tfrac{1}{2} - \tfrac{1}{2} + 2\tfrac{1}{2} - 3 \quad \text{[simplifying]}$$
$$= \tfrac{-1}{2}$$

Example C: $g(x) = (x + 4)(x - 2)$. Solve the equation $g(x) = 0$.

Solution:

$$g(x) = 0$$
$$\therefore (x + 4)(x - 2) = 0$$
$$\therefore \text{ either } x + 4 = 0 \text{ or } x - 2 = 0$$
$$\therefore x = -4 \text{ or } x = 2 \quad \text{[solving } x + 4 = 0 \text{ and } x - 2 = 0\text{]}$$

Example D: $P(x) = (x - 1)(x + 2)(x - 4)$
Solve the equation $P(x) = (x + 2)(x - 4)$.

Solution:

$$P(x) = (x + 2)(x - 4)$$
$$\therefore (x - 1)(x + 2)(x - 4) = (x + 2)(x - 4)$$
$$\therefore (x - 1)(x + 2)(x - 4) - (x + 2)(x - 4) = 0 \quad \text{[subtracting } (x + 2)(x - 4)\text{]}$$
$$\therefore (x + 2)(x - 4)[(x - 1) - 1] = 0 \quad \text{[factorising]}$$
$$\therefore (x + 2)(x - 4)(x - 2) = 0 \quad \text{[simplifying } (x - 1) - 1\text{]}$$
$$\therefore x = -2 \text{ or } x = 4 \text{ or } x = 2$$

Exercise 8b: Substitution, solving

1. $g(x) = (x + 2)(x - 1)$
   a. Find i. g(2) ii. g(4) iii. g(-1) iv. g(-2) v. $g(1\frac{1}{2})$
   b. Solve $g(x) = 0$
   c. Express $g(x)$ in expanded form.

2. $P(x) = 2x^2 - 9x + 4$
   Find a. P(1) b. P(-2) c. $P(\frac{2}{3})$ d. $P(1\frac{1}{3})$ e. P(-2.2)

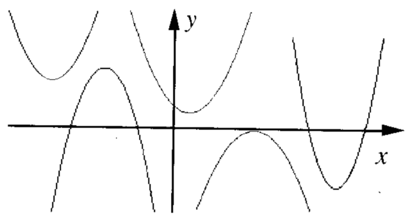
3. $h(x) = (x - 1)(x + 4)$, $g(x) = (x + 3)(x + 4)$, $J(x) = (3x + 2)(x - 1)$
   a. Find i. h(2) ii. h(-3) iii. $h(\frac{1}{2})$ iv. $h(2\frac{1}{3})$ v. $h(-1\frac{1}{9})$
   b. Find i. g(0) ii. $g(\frac{-1}{2})$ iii. g(4)
   c. Find i. J(3) ii. J(-2) iii. $J(\frac{-1}{3})$
   d. Solve i. $h(x) = 0$ ii. $g(x) = 0$ iii. $J(x) = 0$

4. $f(x) = (x - 1)(x - 2)(x - 4)$
   $g(x) = (x - 1)(x - 2)(x + 1)$
   $h(x) = (x - 2)(x + 1)(x + 3)$

   a. Find: i. f(1) ii. $f(2\frac{1}{2})$ iii. g(3) iv. h(2.3) v. h(-3.2)
   b. Solve these equations:
      i. $f(x) = 0$ ii. $g(x) = 0$ iii. $h(x) = 0$
      iv. $f(x) + g(x) = 0$ v. $f(x) - g(x) = 0$ vi. $f(x) - h(x) = 0$
      vii. $(x + 2) f(x) = 0$ viii. $f(x) = 4(x - 2)(x - 1)$

Quadratic Functions

A **quadratic function** has the form $y = Ax^2 + Bx + C$ where A, B and C are **constant** real numbers. A quadratic function is a **polynomial** of **degree 2**. The graphs of quadratic functions are always **parabolas**. This means the graphs will be similar to those shown.



Example E: The functions $y = 2x^2 - 3x + 4$, $y = 5 - \frac{x^2}{2}$ and $y = \frac{2}{3}(x - 1)^2$ are all quadratic because they are of the type $y = Ax^2 + Bx + C$ (or they can be **rearranged** into an equation of the type $y = Ax^2 + Bx + C$).

A **quadratic equation** is of the type $Ax^2 + Bx + C = 0$ or is an equation which can be rearranged into this type.

Example F: The equations $3x^2 - 5x + 1 = 0$, $3(x + 1)^2 - 2 = 0$ and $4x^2 - 5x = 2x^2 - 11x + 4$ are quadratic equations.

Solution of Quadratic Equations

Quadratic equations can sometimes be solved by factorising and where factorisation is possible this method should be used.

Example G: Solve the equation $x^2 + 4x + 3 = 0$

Solution:
$x^2 + 4x + 3 = 0$
$\therefore (x + 3)(x + 1) = 0$ [factorising]
$\therefore x + 3 = 0$ or $x + 1 = 0$
$\therefore x = -3$ or $x = -1$ [solving $x + 3 = 0$ and $x + 1 = 0$]

Example H: Solve the equation $x^2 + 5x = 4x + 2$.

Solution:
$x^2 + 5x = 4x + 2$
$\therefore x^2 + x = 2$ [subtracting $4x$]
$\therefore x^2 + x - 2 = 0$ [subtracting 2]
$\therefore (x + 2)(x - 1) = 0$ [factorising]
$\therefore x = -2$ or 1 [solving $x + 2 = 0$ and $x - 1 = 0$]

Example I: Solve $3x^2 + x - 4 = 0$

Solution:
$3x^2 + x - 4 = 0$
$\therefore (3x + 4)(x - 1) = 0$ [Factorising]
$\therefore x = -1\frac{1}{3}$ or $x = 1$ [Solving $3x + 4 = 0$ and $x - 1 = 0$]

Exercise 8c: Solution of Quadratic Equations by Factorisation

1. $A^2 - 3A - 4 = 0$
2. $A^2 + 3A - 10 = 0$
3. $B^2 - 2B - 8 = 0$
4. $C^2 + 13C + 36 = 0$
5. $A^2 - 3A - 18 = 0$
6. $B^2 - 16 = 0$
7. $x^2 = x + 12$
8. $U^2 + 6U = 8 - U$
9. $2Z^2 + 11Z = Z^2 + 9Z$
10. $5R^2 - 3R + 4 = 4R^2 + R + 16$
11. $(x + 2)^2 = 9$
12. $3x^2 + 2x - 1 = 0$
13. $4x^2 + 8x + 3 = 0$
14. $\frac{3}{x + 1} = x - 1$
15. $(x + 1)^2 - 2(x + 1) - 3 = 0$

Solution of Quadratic Equations by Formula

Where a quadratic equation cannot be factorised directly there is a formula available which can be used to solve those quadratic equations which have solutions.

The solution to the general quadratic equation, $Ax^2 + Bx + C = 0$ is:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Example J: Use the result above to find the solutions to the equation $2x^2 + 8x - 3 = 0$ correct to 2 decimal places.

Solution: For the equation $2x^2 + 8x - 3 = 0$, A = 2, B = 8, C = -3.

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 2 \times -3}}{4} \quad \text{[substituting into } x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\text{]}$$

$$\therefore\ x = \frac{-8 \pm \sqrt{88}}{4}$$

$$\therefore\ x = \frac{-8 + \sqrt{88}}{4} \text{ or } \frac{-8 - \sqrt{88}}{4}$$

$$\therefore\ x = \frac{-8 + 9.38}{4} \text{ or } \frac{-8 - 9.38}{4}$$

$$\therefore\ x = 0.35 \text{ or } -4.35$$

Sometimes the quadratic equation may need to be rearranged into the form $Ax^2 + Bx + C = 0$ as the following example shows.

Example K: $(x + 1)^2 = x + 5$

Solution:

$(x + 1)^2 = x + 5$

$\therefore\ x^2 + 2x + 1 = x + 5$ [expanding $(x + 1)^2$]

$\therefore\ x^2 + x - 4 = 0$ [gathering all terms together]

$\therefore\ x = \frac{-1 \pm \sqrt{1^2 - 4.1.\text{-}4}}{2}$ [substituting A = 1, B = 1, C = -4]

$\therefore\ x = \frac{-1 \pm \sqrt{17}}{2}$

$\therefore\ x = 1.6, -2.6$

Exercise 8d: Quadratic Equations

1. Solve each of these equations giving answers correct to two decimal places:
   a. $x^2 + 4x + 1 = 0$ b. $x^2 + 8x + 3 = 0$ c. $2x^2 - 7x + 1 = 0$
   d. $3x^2 + 6x - 7 = 0$ e. $2R^2 + R - 8 = 0$ f. $x^2 + 1.1x - 0.3 = 0$
   g. $2.3x^2 - 3.5x - 1.2 = 0$ h. $3A^2 + 5A = 2A^2 - 3A + 2$
   i. $8U^2 + 3U + 5 = 2U^2 + 11U + 7$ j. $\frac{1}{3}x^2 + 1\frac{1}{2}x + \frac{1}{12} = 0$

2. Solve each of these equations to two decimal places:
   a. $(x + 3)^2 = 8$ b. $(2x - 3)^2 = 7$ c. $3(x - 1)^2 + 1 = 6$
   d. $x + 1 = \frac{3}{x}$ e. $(2x + 1)^2 = x^2 - x + 5$ f. $\frac{x+1}{3} + \frac{1}{x} = 2$
   g. $\frac{2x+1}{x+3} = \frac{x+4}{x+1}$ h. $3(x + 1)^2 - 6(x + 1) + 2 = 0$
   i. $\frac{x+1}{x^2+x+1} = 0.2$ j. $\frac{x^2+x+1}{x^2-x+1} = 3$

Solution of Quadratic Equations using a Programmable Calculator

Using a programmable calculator makes the quick solution of large numbers of quadratic equations possible. All that is required are the values of A, B and C for the formula $\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$.

Using the Casio *fx* series, the program:

? → A : ? → B : ? → C : (-B + √ (B² − 4AC)) ÷ (2A) ◢
(-B − √ (B² − 4AC)) ÷ (2A) ◢

will generate the roots of any quadratic equation, provided they are real numbers. Owners of other types of calculators will need to consult their manuals to get an equivalent program.

Word Problems

The solution of many word problems can be found using a quadratic equation.

Example L: The length of a small room is 3m more than its width. It has an area of $72m^2$. Find the width.

Solution: Let the width be w

$\therefore$ the length is $w + 3$ [length is 3 metres more than width]

$\therefore\ w(w + 3) = 72$ [width × length = area]

$\therefore\ w^2 + 3w = 72$ [expanding]

$\therefore\ w^2 + 3w - 72 = 0$ [rearranging]

$\therefore\ w = 7.12$ or -10.12 [solving the quadratic equation using A = 1, B = 3, C = -72]

$\therefore$ the width is 7.12 metres [reject -10.12 metres because it is impossible to have a negative length]

Exercise 8e: Word Problems

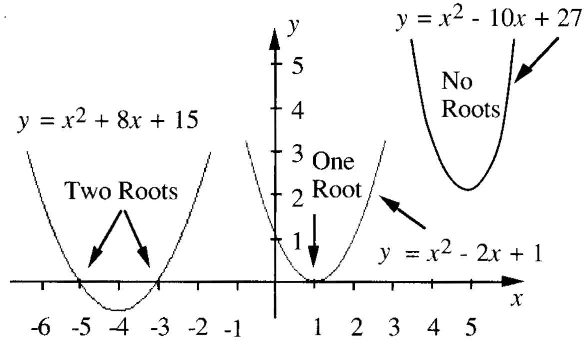
For each of these problems write an equation then solve.

1. A room is 2m longer than it is wide. It has an area of $100m^2$. Find the width and the length.
2. A room is twice as long as it is wide. It has an area of $200m^2$. Find its width.
3. A triangle has a base which is 2 metres longer than its height. It has an area of $10m^2$. Find the height of the triangle.
4. A triangle has a base which is 4 metres longer than its height. Its area is $15m^2$. Find the base.

5. John is 29 years older than Peter. The product of their ages is 1 272. Find their ages.
6. The product of two consecutive natural numbers is 3 782. Find the numbers.
7. A man has a garden with an area of 100m². The garden is in the shape of a square. He increases all the sides of this garden by a certain amount and now has an area of 150m². How much did he increase each side by?
8. The sum of the area and perimeter of a square gives a value of 15. Find the perimeter of the square.
9. A box is constructed so that its length is one metre more than its height which in turn is one metre more than its width. It has a total surface area of 16m². Find the lengths of all sides.
10. Find any number which when added to its reciprocal gives 5.
11. The sum of the first n natural numbers is $\frac{1}{2}n(n+1)$.
    a. Find how many natural numbers you have to add to get a sum of 496.
    b. How many do you have to add to exceed 1 000?
12. The height of a right-angled triangle is one metre more than its base. The hypotenuse of the triangle is 4 m. Find the lengths of the base and height.
13. A girl is asked by her friend how much she gets paid per hour. She answers in the following way: "the number of hours I work each week is one more than double my hourly pay. I get paid $153 per week. Work it out yourself." How much did she get paid per hour?

Nature of the Roots of a Quadratic Equation

The roots of the equation $Ax^2 + Bx + C = 0$ are the x **intercepts** of the parabola $y = Ax^2 + Bx + C$ (the points where the graph cuts the x axis). The sketch shows several quadratic functions and their roots and shows that quadratic equations can have *two*, *one* or *no* real roots.



The **discriminant** of the equation $Ax^2 + Bx + C = 0$ is:

$$B^2 - 4AC$$

The discriminant can be used to check the number of roots that a particular quadratic equation has. The equation $Ax^2 + Bx + C = 0$ will have:

Two distinct real roots if $B^2 - 4AC > 0$
One real root if $B^2 - 4AC = 0$
No real roots if $B^2 - 4AC < 0$

The reason for this result is that:

i. if $B^2 - 4AC > 0$ then there are two roots, namely $\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$.
ii. If $B^2 - 4AC = 0$, the root will be $x = \frac{-B}{2A}$ by substituting into the formula.
iii. If $B^2 - 4AC < 0$ then the discriminant is negative and the roots do not exist.

The **nature** of the roots refers to how many roots an equation has and whether or not the roots are real.

Example M: For each equation find the discriminant and the nature of the roots:

a. $9x^2 + 3x + 0.25 = 0$ b. $x^2 + x + 1 = 0$ c. $x^2 + 4x + 1 = 0$

Solution:

a. Discriminant $= B^2 - 4AC$
$= 3^2 - 4 \times 9 \times 0.25$
$= 9 - 9$
$= 0$, one real root

b. Discriminant $= B^2 - 4AC$
$= 1^2 - 4 \times 1 \times 1$
$= 1 - 4$
$= -3$, no real roots

c. Discriminant $= B^2 - 4AC$
$= 4^2 - 4 \times 1 \times 1$
$= 16 - 4$
$= 12$, two real roots

Example N: Find k so that $2x^2 - 3x + k = 0$ has exactly one root.

Solution: If the equation has exactly one root then the discriminant $B^2 - 4AC = 0$.

$\therefore (-3)^2 - 4.2.k = 0$
$\therefore 9 - 8k = 0$
$\therefore 9 = 8k$
$\therefore k = 1.125$

Exercise 8f

1. For each of the following equations:
   i. find the discriminant, and ii. state the nature of the roots.
   a. $2x^2 + 3x + 1 = 0$ b. $4x^2 - 5x + 2 = 0$
   c. $U^2 + U + 4 = 0$ d. $a^2 + 8a + 16 = 0$
   e. $2x^2 - 2x + 3 = 0$ f. $3x^2 + x = x^2 + 2x - 3$
   g. $2x^2 + 4x = 5 - x - x^2$ h. $(2x + 1)^2 = 2x - 3$
   i. $x + \frac{1}{x} = 2$ j. $2x - \frac{3}{x} = 4$

2. a. Find k so that $kx^2 - 2x + 4 = 0$ has one root.
   b. Find P so that $3x^2 - 8x - P = 0$ has one root.
   c. Find q so that $4x^2 + 3qx + 2 = 0$ has exactly one root.
   d. Find all values of k so that $kx^2 - 5x + 6 = 0$ has two roots.
   e. Find all values of p so that $x^2 - 3x + p = 0$ has no roots.

Solution of Cubic and Higher Degree Equations

Although there are formulae for solving cubics and some higher degree polynomial equations, they are not studied in this course. Despite this, it is possible to get approximate solutions which are as accurate as the reader may wish.

Example O: Solve $x^3 - 2x^2 - 5x + 7 = 0$ to one decimal place and sketch the graph of $y = x^3 - 2x^2 - 5x + 7$.

Solution: i. Using a graphical calculator (Casio *fx*) it would be necessary to draw the graph then using the TRACE application find where the graph cuts the x axis.

ii. Using a programmable calculator (Casio *fx*), a suitable program is:

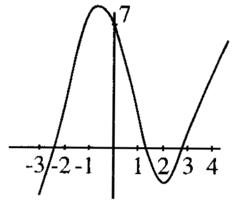
$$? \rightarrow x : x^3 - 2x^2 - 5x + 7$$

Starting from a negative integral value keep on inputting steadily increasing values for x. Record the y values obtained.

A table of (x, y) values is established. (The x values in tables should not exceed the magnitude of the term independent of x. In this case it is 7.)

x	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
y	-399	-251	-143	-69	-23	1	9	7	1	-3	1	19	57	121	217

If a polynomial changes its sign in passing from one x value to the next then there is a root between those x values. Here, there are roots for the equation $x^3 - 2x^2 - 5x + 7 = 0$ between -3 and -2, between 1 and 2, between 2 and 3. A sketch of the graph appears to the right.



The root between 1 and 2 is closer to 1 than it is to 2 because the y value ($y = 1$) when $x = 1$ is smaller in size than the y value ($y = -3$) when $x = 2$ which has size 3. In order to find the root correct to 1 decimal place successive values for x of 1.1, 1.2, 1.3... are substituted into the formula until the sign changes, resulting in the following table:

x	1	1.1	1.2	...
y	1	0.411	-0.152	...

As the y value for $x = 1.2$ is smaller in magnitude than the y value for $x = 1.1$ we deduce the root is closer to 1.2. Hence the root correct to 1 decimal place is 1.2.

If the reader wishes, s/he can repeat this process to verify that the root correct to 2 decimal places is 1.17.

Exercise 8g

1. Which of the values -1, 1, 2, 3 is the best approximation to the only root of the equation $x^3 - x - 7 = 0$?
   [**Hint:** see which value when substituted into $x^3 - x - 7 = 0$ gives a value closest to 0.]

2. The equation $x^3 + 3x^2 + 2x + 1 = 0$ has a root which lies between -3 and -2. Which of the following is the best approximation to that root?
   i. -3 ii. -2.5 iii. -2.1 iv. -2 v. 2

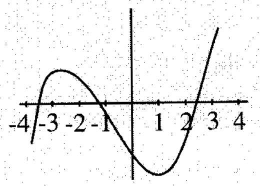
3. a. Prove the equation $x^3 + x^2 - 5x - 1 = 0$ has a root which lies between -1 and 0.
   b. Find that root correct to one decimal place.

4. $y = x^3 - 2x^2 - x + 3$
   a. Locate the root(s) of the equation $x^3 - 2x^2 - x + 3 = 0$ correct to the nearest integer.
   b. Sketch the graph of $y = x^3 - 2x^2 - x + 3$.

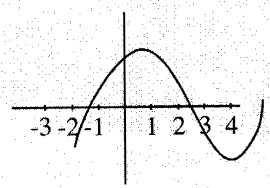
5. Repeat the procedure used in question 4 on:
   a. $y = x^3 - 6x^2 + 11x - 7$
   b. $y = x^3 - 7x + 4$
   c. $y = x^3 - 2x^2 - 5x + 3$
   d. $y = x^4 - 6x^3 + 7x^2 + 6x - 7$

*6. The graphs shown below can each be represented by equations of the type $y = x^3 + Ax^2 + Bx + c$. Find such equations for each graph.

a.



b.



Problems and Investigations

A box has a base whose length is always twice its width. The sum of its height, its width and its length is always less than or equal to 30cm. The length, the width and the height are all whole numbers.

a. Find a formula for the volume in terms of the width.
b. Find a formula for the surface area.
c. By use of a spreadsheet or otherwise find the dimensions of the box for:
   i. maximum surface area.
   ii. maximum volume.

9. CO-ORDINATE GEOMETRY OF THE STRAIGHT LINE

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 GEOMETRY

- *to find the distance between 2 points on a co-ordinate plane*
- *given an equation representing a straight line in 2 dimensions, to find its gradient*

The Distance between Two Points

To find the distance between two points a right-angled triangle is used in which the line joining two points forms the **hypotenuse.** The distance required is found using **Pythagoras' theorem.**

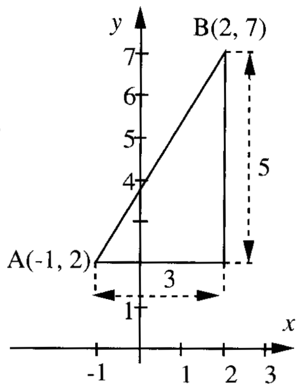
Example A: The distance between A (-1, 2) and B (2, 7) is the length of the hypotenuse on the triangle shown.

From the diagram:

a. The *horizontal* displacement of A to B is 3.
b. The *vertical* displacement of B from A is 5.

By Pythagoras' theorem:

$$AB^2 = 3^2 + 5^2$$
$$= 9 + 25$$
$$= 34$$
$$\therefore AB = \sqrt{34}$$
$$= 5.83 \text{ (2 dp)}$$



The method shown above can be generalised:

The distance, d, between the points with co-ordinates (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Example B: The distance between (2, 3) and (-1, 4) is found by letting (x_1, y_1) be (2, 3), and (x_2, y_2) be (-1, 4) and substituting.

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(2 - {}^-1)^2 + (3 - 4)^2} \qquad \text{[substituting]} \\ &= \sqrt{3^2 + (-1)^2} \\ &= \sqrt{10} \\ &= 3.16 \text{ (2 dp)} \end{aligned}$$

A variety of problems can be solved using the distance between a pair of points.

Example C: Show that (1, 2), (2, 6) and (6, 5) are the vertices of an **isosceles** triangle.

Solution: In an isosceles triangle two of the sides are of equal length.

Length of line joining (1, 2) and (2, 6) is $\sqrt{(2-1)^2+(6-2)^2} = \sqrt{17}$

Length of line joining (1, 2) and (6, 5) is $\sqrt{(6-1)^2+(5-2)^2} = \sqrt{34}$

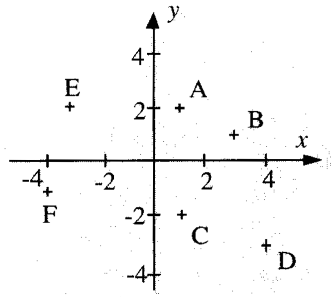
Length of line joining (2, 6) and (6, 5) is $\sqrt{(6-2)^2+(5-6)^2} = \sqrt{17}$

Two of the lines are of equal length and thus the triangle is isosceles.

Exercise 9a: Distance between Points

1. Find the distance between each of the following pairs of points:
   a. (-1, 2) and (3, 6) b. (8, 5) and (2, -3)
   c. (-3, 5) and (2, 17) d. (-1, 4) and (-3, -2)
   e. (2, -5) and (-2, 0)

2. Examine the diagram and find the lengths of each of the following line segments:
   a. AB
   b. EB
   c. BC
   d. ED
   e. AD



3. a. Draw the triangle A(5, 1), B(0, 2) and C(4, 6).
   b. Show that this triangle is isosceles but not equilateral.

4. Show that the triangle with vertices (1, 1), (7, 3) and (3, 7) is isosceles.

5. Which of the points (0, 1), (-1, 2) or (-2, 5) forms an isosceles triangle with (6, 4) and (1, 9)?

6. Show that each of the following ordered pairs form right-angled triangles:
   a. (2, 3), (-2, 1) and (1, 5)
   b. (-1, 4), (3, -2) and (2, 6)
   c. (1, 3), (-4, 2) and (3, -7)

7. a. Find x so that the distance between (x, 3) and (-1, 7) is 5.
   b. Find x so that the distance between (x, -4) and (5, 1) is 13.

8. Find y so that the points (-1, 3), (2, y) and (3, -3) form a right-angled triangle. [y is a whole number.]

9. Find x so that the points (x, 2), (2, 0) and (-2, -2) form a right-angled triangle.

10. Show that the points A(0, -4), B(2, 3), C(8, 4) and D(7, -2) form a kite. Find its area.

11. A tramper is 14 km West and 15 km North of base at the end of the first day's tramping. At the end of the second day, the tramper is 8 km East and 25 km North of base. How far did the tramper travel during the second day?

12. After a day's sailing since leaving an island the yacht *Neville* is 40 km West and 65 km North of the island. At the same time, the yacht *Cosmo* is 32 km East and 75 km North of the island. What is the distance between them?

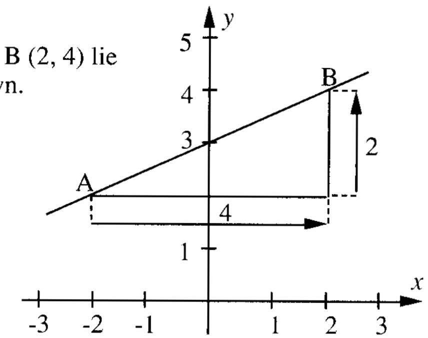
The Gradient of a Straight Line

For any pair of points on a non-vertical straight line, the ratio $\dfrac{\text{vertical change}}{\text{horizontal change}}$ has a constant value and is called the **gradient** (or **slope**) of the line. Gradient has the symbol **m**.

$$m = \frac{\text{vertical change}}{\text{horizontal change}}$$

Note: If a line is vertical its gradient is undefined.

Example D: The points A (-2, 2) and B (2, 4) lie on the straight line shown.



From the diagram, going *from* A *to* B, the *vertical* change is 2 and the *horizontal* change is 4. The gradient of AB, the straight line going through A and B, is:

$$\frac{\text{vertical change}}{\text{horizontal change}} = \frac{2}{4} = \frac{1}{2}$$

By convention:

a. a vertical change *upward is positive* and *down is negative*.

b. a horizontal change *to the right is positive* and *to the left is negative*.

Note: The gradient in example D can be calculated by going *from B to A* instead of A to B. In this case:

- the vertical change is -2. [going down]
- and the horizontal change is -4. [going to the left]

The gradient of the line is now:

$$\frac{\text{vertical change}}{\text{horizontal change}} = \frac{-2}{-4} = \frac{1}{2}\text{, the same value as before.}$$

The procedures in example D can be used to show that the gradient of the line passing through the points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

[**Note:** The gradient of a straight line is the rate at which the y values change as x increases by 1.]

Example E: Find the gradient of the line joining (2, -3) and (-3, 2).

Solution: Let (x_1, y_1) be (2, -3) and (x_2, y_2) be (-3, 2).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2 - {-3}}{-3 - 2} \quad \text{[substituting]}$$

$$= \frac{5}{-5}$$

$$= -1$$

Note: i. It does not matter which point is (x_1, y_1) and which is (x_2, y_2).

ii. If the point (a, b) is on a line with gradient $\frac{k}{1}$ then so is the point (a + 1, b + k)

Example F: Find k so that the line joining (3, k) and (5, 6) has gradient 4.

Solution: i. Algebraic; Let (x_1, y_1) be (3, k) and (x_2, y_2) be (5, 6). The gradient of the line joining (3, k) and (5, 6) is $\frac{y_2 - y_1}{x_2 - x_1} = 4$

$$\therefore \frac{6 - k}{5 - 3} = 4 \quad \text{[substituting]}$$

$$\therefore \frac{6 - k}{2} = 4$$

$$\therefore 6 - k = 8 \quad \text{[multiplying by 2]}$$

$$\therefore k = -2 \quad \text{[solving for k]}$$

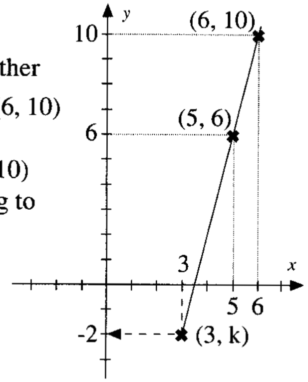
ii. Graphical; i. Plot (5, 6)

ii. Use the gradient $4 = \frac{4}{1}$ to get another point on the line (5 + 1, 6 + 4) = (6, 10) [see previous notes]

iii. Draw the line through (5, 6), (6, 10)

iv. Read k, the y value corresponding to $x = 3$, off the line

v. k = -2



Exercise 9b: Gradient

1. For each of the following pairs of points find the gradient of the line which goes through them.

a. (-1, 2), (2, 3)	b. (2, 5), (6, 8)
c. (-2, 1), (0, 3)	d. (-5, 2), (-6, 4)
e. (3, 2), (3, 4)	f. (4, 2), (3, 2)

2. Show that the line going through (1, 3) and (3, 7) has the same gradient as the line going through (2, 8) and (4, 12).

3. Show that the line going through (0, 2) and (2, 8) is parallel to the line going through (1, -1) and (3, 5).

4. Prove that the line going through (2, 5) and (3, 6) has a different gradient to the line going through (-2, 3) and (1, 12).

5. Find out if the line going through (-1, 2) and (2, 8) is parallel to the line going through (-2, 2) and (3, 8). [**Note:** if 2 lines are **parallel** they have the same gradient.]

6. Find out if the line going through (3, 2) and (4, 0) is parallel to the line going through (1, 3) and (-1, 7).

7. Find k so that the line joining (2, k) and (3, 5) has gradient 2.

8. Find k so that the line joining (3, k) and (5, 8) has gradient 2.

9. Find k so that the line joining (-1, 5) and (3, k) has gradient -2.

10. Find k so that the line joining (-2, 6) and (3, k) has gradient 1.

11. Find x so that the line joining (x, 3) and (5, 8) has gradient $\frac{5}{4}$.

12. Find x so that the line joining (5, 8) and (x , 12) has gradient $\frac{1}{2}$.

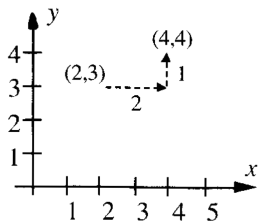
13. Find x so that the line joining (2, 5) and (x, 10) has gradient $\frac{-1}{3}$.

14. (a, b) lies on a straight line having gradient $\frac{2}{3}$. If we increase a by 6 what does b increase by?

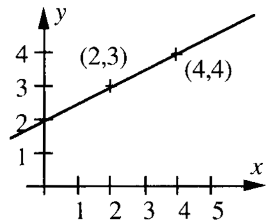
15. (a, b) lies on a straight line having gradient -2. If we increase a by 2 what does b change by?

Drawing a Line from its Gradient and a Point

Example G: The straight line going through the point (2, 3) with gradient $\frac{1}{2}$ is drawn as follows:



Plot the point (2, 3). Since the gradient is $\frac{1}{2}$, another point on the line can be found by making a horizontal change of 2 and a vertical change of 1 from (2, 3).



This gives the point (4, 4). The straight line is now drawn through (2, 3) and (4, 4).

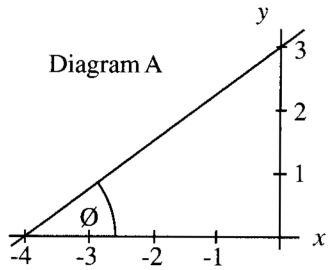
Exercise 9c

1. Draw the straight line going through (0, 1) with gradient $\frac{2}{5}$.
2. Draw the straight line going through (-2, 4) with gradient $\frac{3}{4}$.
3. Draw the straight line going through (3, 2) with gradient $\frac{1}{4}$.
4. Draw the straight line going through (4, 3) with gradient $\frac{-2}{3}$.

5. Draw the straight line going through (2, 1) with gradient $\frac{-1}{2}$.
6. Draw the straight line going through (-2, -2) with gradient 0.
7. Draw the straight line going through (-3, 0) with gradient undefined.

Relationship between Angle and Gradient

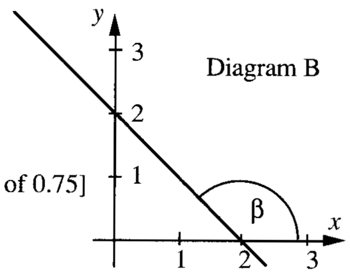
If the angle marked ø in Diagram A is measured with a protractor, its size to the nearest degree will be 37°. Similarly, the size of the angle marked ß in Diagram B is 135°.



In Diagram A, the line passes through the points (-4, 0) and (0, 3) and hence has gradient $\frac{3-0}{0-{}^-4} = 0.75$

The tangent of the angle ø is:

$$\tan ø = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{3}{4} \quad \text{[substituting]}$$
$$= 0.75$$
$$\therefore\ ø = 36.9° \quad \text{[finding inverse tangent of 0.75]}$$



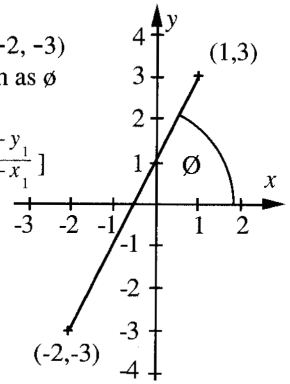
For Diagram B, the gradient is $m = \frac{2-0}{0-2} = -1$

Add 180º to $\tan^{-1}(-1)$ to get the angle: -45 + 180 = 135°

These results can be generalised:

> The *tangent of the angle* a straight line makes with the positive horizontal direction is equal to its *gradient*, ie tan ø = gradient where ø is an angle with positive horizontal direction.

Example J: The angle that the line going through (-2, -3) and (1, 3) makes with the positive horizontal is shown as ø in the diagram.



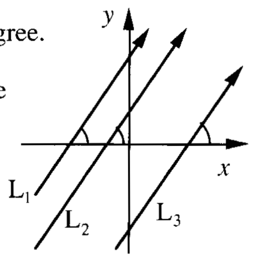
The gradient of the line is $m = \frac{3-{}^-3}{1-{}^-2}$ [using $m = \frac{y_2-y_1}{x_2-x_1}$]

$$= \frac{6}{3}$$
$$= 2$$
$$\therefore\ \tan ø = 2$$
$$\therefore\ ø = 63.4°$$

'tanø = 2' is solved by calculating:

[2] [inv] [tan]

Note: A protractor will give ø as 63°, to the nearest degree.



Parallel lines have the same gradient because they make the same angle with the positive horizontal direction. In the diagram, L_1, L_2 and L_3 are parallel lines. The angles they make with the positive horizontal are equal because they are corresponding angles. The tangents of those angles are the same and so the gradients are equal.

Exercise 9d: Gradient

1. For lines with the following gradients find the angle each makes with the positive horizontal direction.
   a. $\frac{2}{5}$ b. $\frac{3}{4}$ c. $\frac{1}{4}$ d. $-\frac{2}{3}$ e. $-\frac{1}{2}$ f. 0 g. undefined

2. For lines going through the following pairs of points find the angle each makes with the positive horizontal direction.
   a. (-1, 3), (2, 4) b. (4, 5), (8, 8)
   c. (-3, 1), (-1, 3) d. (-5, 1), (-6, 3)
   e. (4, 3), (4, 5) f. (4, 1), (3, 1)

3. Find the gradients of lines making the following angles with the positive horizontal direction:
   a. 56.3° b. 60.25°
   c. 53.13° d. 24.23°
   e. 146.98°

Problems and Investigations

1. a. Investigate trios of points of type $(x, 3x)$, $(\frac{1}{3x}, \frac{1}{x})$, $(3x, 9x)$ for different values of x.
   b. What conclusion might you reach about such trios of points?
   c. Prove your conclusion.

2. Investigate the claim that all lines joining points (x, y) and (y, x) are parallel so long as x and y are different.

3. Investigate the claim that the line joining (1, 1) and (t, t^2) always makes an angle greater than 45° with the positive horizontal direction if $t > 0$.

4. Is it true that the points (a, b), (c, d) and (ta + (1 – t) c, tb + (1 – t) d) always lie on the same line?

10. EQUATION OF A STRAIGHT LINE

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 5 ALGEBRA

- *to graph linear rules and interpret the slope and intercepts on an integer co-ordinate system*

LEVEL 7 GEOMETRY

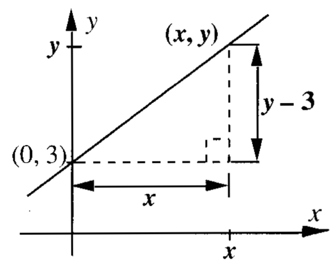
- *given an equation representing a straight line in 2 dimensions, to find its gradient*
- *to write and use equations of straight lines in 2 dimensions, given necessary and sufficient information*

Introduction

The **equation of a straight line** can be found using various methods.

Example A: A straight line goes through (0, 3) with gradient $\frac{3}{4}$. If (x, y) is a point on the line, a right-angled triangle can be drawn as shown in the diagram.

From the diagram, the *horizontal* change from (0, 3) to (x, y) is x.
The *vertical* change from (0, 3) to (x, y) is $y - 3$.



$\therefore$ gradient $= \frac{y-3}{x}$

$= \frac{3}{4}$

$\therefore \frac{y-3}{x} = \frac{3}{4}$ [as gradient $= \frac{3}{4}$]

$\therefore y - 3 = \frac{3}{4}x$ [multiplying by x]

$\therefore y = \frac{3}{4}x + 3$ [adding 3]

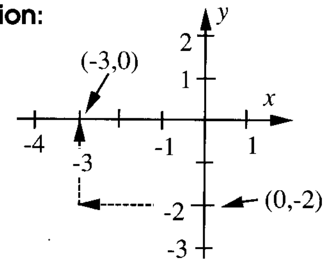
Generally the equation of any straight line can be written in the form:

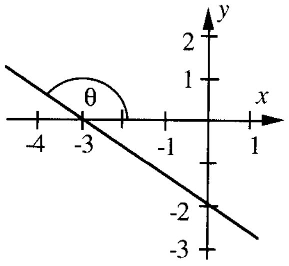
$y = mx + c$ where m is the gradient and c is the y intercept.

The graph of a straight line can be drawn from the equation of the line.

Example B: To draw the line $y = \frac{-2}{3}x - 2$ and find the angle it makes with the positive horizontal direction, proceed as follows.

Solution:





The y intercept (0, -2) is plotted. From the gradient $\frac{-2}{3}$, a horizontal change of -3 and a vertical change of 2 take us to the point (-3, 0).

A straight line is drawn through the two points. The angle required is labelled.

Note: A horizontal change of 3 and a vertical change of -2 could have been used.

The angle required is marked θ and has a tangent of $-\frac{2}{3}$. Most calculators give $\theta = \tan^{-1}\left(-\frac{2}{3}\right)$ as -33.7° (to one decimal place). Adding 180° to -33.7° gives 146.3°, the angle made with the *positive horizontal direction.*

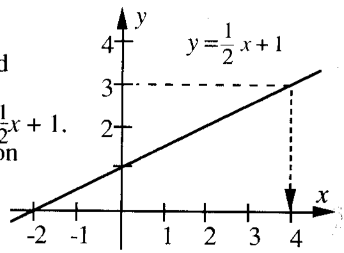
Note: If the line has a positive gradient, the line will make an acute angle with the positive horizontal direction and the calculator will give the angle in one step.

The equation of a *vertical line* is of the type $x = k$, where k is the x intercept.
The equation of a *horizontal line* is of the type $y = c$, where c is the y intercept.

Example C: Find the value of k so that (k, 3) lies on the line $y = \frac{1}{2}x + 1$.

Solution:

i. Graphical: Draw the graph of $y = \frac{1}{2}x + 1$ and locate 3 on the y axis.
From 3 on the y axis, go across to the line $y = \frac{1}{2}x + 1$.
Now go down to the x axis and read the solution which is $x = 4$. Hence k = 4.



ii. Algebraic: Substitute $x = k$ and $y = 3$ (obtained from the co-ordinates of the point) directly into the equation. This gives:

gives:	$3 = \frac{1}{2}k + 1$	
	$2 = \frac{1}{2}k$	[subtracting 1]
	$k = 4$	[multiplying by 2 and rearranging]

Exercise 10a: Straight Line Equations

1. Draw graphs of each of the following straight lines:
   a. $y = 2x + 1$ b. $y = 3x - 4$ c. $y = \frac{x}{4} + 1$
   d. $y = \frac{2}{3}x + 2$ e. $y = -x + 5$ f. $y = \frac{-3}{4}x + 3$
   g. $y = 3 - \frac{2}{5}x$ h. $y = 2$ i. $x = -3$
   j. $y + 5 = 0$

2. For each line in question 1, find the gradient and the angle with the positive horizontal direction.

3. a. (2, k) lies on the graph of $y = 2x + 1$. Find k.
   b. (-3, k) lies on the graph of $y = 3x$. Find k.
   c. (2, k) lies on the graph of $y = 3$. Find k.
   d. (k, 5) lies on the graph of $y = 3x + 2$. Find k.
   e. (k, -6) lies on the graph of $y = 2x + 2$. Find k.
   f. (k, -8) lies on the graph of $y = 3x + 1$. Find k.
   g. $(1\frac{1}{3}, k)$ lies on the graph of $y = \frac{x+5}{2}$. Find k.
   h. (d, 3) lies on the graph of $y = 3x + 1$. Find d.
   i. (r, -2) lies on the graph of $y = \frac{2}{3}x - 1$. Find r.
   j. $(p, -1\frac{1}{2})$ lies on the graph of $y = \frac{1}{3}x + 1$. Find p.

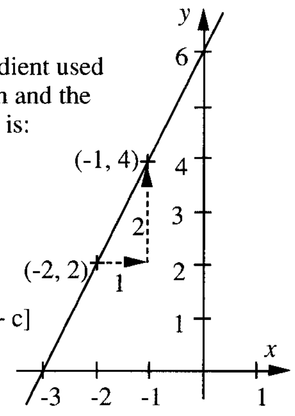
4. Find the x and y intercepts of the following straight lines:
   a. $y = 2x - 4$ b. $y = 3x + 6$ c. $y = \frac{2}{3}x - 4$
   d. $y = -x + 2$ e. $y = 5 - \frac{2}{3}x$

The Equation of a Straight Line when the Gradient and a Point on the Line are Known

Example D: Find the equation of the line going through (-2, 2) with gradient 2.

Solution:

i. Graphical: The point (-2, 2) is plotted and the gradient used to find another point on the line. The line is drawn and the y intercept, 6, is read off. The equation of the line is:
$y = 2x + 6$. [substituting in $y = mx + c$]



ii. Algebraic: Since the gradient is 2, the equation is $y = 2x + c$. The point (-2, 2) lies on the line and so will satisfy the equation:

$y = 2x + c$	
$2 = 2 \times -2 + c$	[substituting (-2, 2) in $y = 2x + c$]
$\therefore 2 = -4 + c$	[simplifying]
$\therefore 6 = c$	[adding 4]

$\therefore$ the equation is $y = 2x + 6$

Using the algebraic method in example D, it can be shown that a straight line with gradient m going through (x_1, y_1) has the equation:

$$y - y_1 = m(x - x_1)$$

If the equation of a straight line has to be found quickly or involves awkward numbers then the formula above is very useful.

Example E: The equation of the straight line going through (3, 5) with gradient -2 is:

$y - 5 = -2(x - 3)$ [substituting into $y - y_1 = m(x - x_1)$]

$\therefore\ y - 5 = -2x + 6$ [expanding]

$\therefore\ y = -2x + 11$ [adding 5]

Exercise 10b

Find the equation of the following straight lines:
1. passes through (1, 2) with gradient 1.
2. passes through (2, 7) with gradient 2.
3. passes through (1, 1) with gradient 3.
4. passes through (-1, -1) with gradient 4.
5. passes through (2, -3) with gradient $\frac{1}{2}$.
6. passes through (3, 4) with gradient $\frac{2}{3}$.
7. passes through (-1, 5) with gradient -2.
8. passes through (4, -1) with gradient $\frac{-3}{4}$.

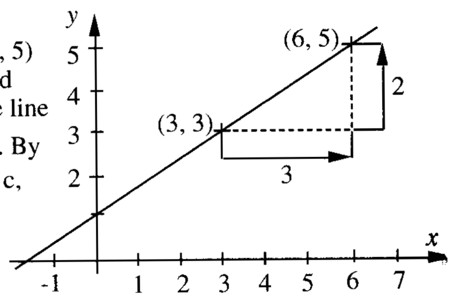
The Equation of a Straight Line going through Two Points

The equation is found by using the two points to find the gradient.

Example F: Find the equation of the line going through (3, 3) and (6, 5).

Solution:

i. Graphical: The points (3, 3) and (6, 5) are plotted. From the horizontal and vertical changes, the gradient of the line is $m = \frac{2}{3}$ and the y intercept is c = 1. By comparison with the line $y = mx + c$, the line is $y = \frac{2}{3}x + 1$



ii. Algebraic: The gradient of the line is:

$m = \frac{5 - 3}{6 - 3}$ [substituting into $\frac{y_2 - y_1}{x_2 - x_1}$]

$= \frac{2}{3}$

The equation of the line is:

$y - 5 = \frac{2}{3}(x - 6)$ [substituting into $y - y_1 = m(x - x_1)$]

$\therefore\ y - 5 = \frac{2}{3}x - 4$ [expanding]

$\therefore\ y = \frac{2}{3}x + 1$ [adding 5]

Using the algebraic method of the above example it is possible to show that a straight line going through the points (x_1, y_1) and (x_2, y_2) has the equation:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Exercise 10c

Find the equation of the line going through:

1. (1, 0) and (2, 1)
2. (2, 7) and (3, 9)
3. (-2, 1) and (2, 4)
4. (-2, 6) and (3, 1)
5. (-6, -8) and (3, -2)
6. (1, 4) and (5, 4)
7. (-2, 1) and (4, -2)
8. (4, -1) and (12, -7)

Other Forms of the Straight Line Equation

The equation of a straight line can be written in different ways, each of which represents the same set of points on a graph.

Example G: The equation $y = \frac{3}{4}x + 2$ is that of a straight line with gradient $\frac{3}{4}$ and y intercept 2. It can be rearranged into a number of other forms, each of which represents the same set of points on a graph.

$y = \frac{3}{4}x + 2$

$\therefore\ 4y = 3x + 8$ [multiplying by 4]

$\therefore\ 4y - 3x - 8 = 0$ [subtracting $3x$ and 8]

Any equation which can be arranged into the form $y = mx + c$ is a straight line.

Example H: $3x - 2y + 5 = 6x - 4$ is the equation of a straight line because it can be rearranged into the form $y = mx + c$:

$3x - 2y + 5 = 6x - 4$

$\therefore\ -2y = 3x - 9$ [subtracting $3x$ and 5]

$\therefore\ y = \frac{-3}{2}x + 4\frac{1}{2}$, which is of the form $y = mx + c$.

Thus $3x - 2y + 5 = 6x - 4$ is the equation of a straight line with gradient $= \frac{-3}{2}$ and y intercept $= 4\frac{1}{2}$.

Exercise 10d

1. Which of the following are equations of straight lines:

   a. $y = \frac{2}{x}$ b. $y = 2x^2 - 3$ c. $y = 3x + 1$
   d. $2x + 3y = 6$ e. $y = |x| + 1$ f. $y = x^3 - 3$
   g. $y = x^2 - 5x$ h. $y = 2y + 3x - 2$ i. $y = 2^x$

2. Each of the following is the equation of a straight line. In each case write it the form: $y = mx + c$.

   a. $3y - 2x = 5$ b. $2y - 3x = -7$ c. $4y - 3x + 5 = 0$
   d. $3y - 2x = y + x - 4$ e. $3y - \frac{2}{3} = 4x + \frac{5}{2}$

3. a. Show that any equation of the form $Ax + By + C = Dx + Ey + F$, where B, C, D, E, and F are real numbers, is that of a straight line.

   b. Which of the following are straight line equations?
   i. $2x^2 + 3y = 2$ ii. $2y - x + 7 = 4x$ iii. $y = 2^x + 3$

Applications

Example I: A tank contains 3 litres of water. More water is run into the tank at a constant rate of 0.5 litres per hour.

a. Fill in the table to show the amount of water in the tank at each time.

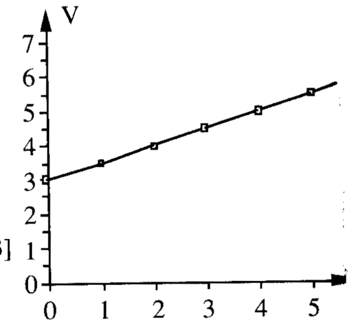
Time (t) in hours	0	1	2	3	4	5
Volume (V) in L	3					

b. Plot V against t and find the equation relating V to t.
c. If the tank can hold 6.3 L, how long it will take to fill the tank?

Solution:

a.

t	0	1	2	3	4	5
V	3	3.5	4	4.5	5	5.5



b. The graph is a straight line with gradient $\frac{1}{2}$ and V intercept 3, hence $V = \frac{1}{2}t + 3$

c. When the tank is full V = 6.3

$\therefore\ 6.3 = \frac{1}{2}t + 3$ [substituting into $V = \frac{1}{2}t + 3$]

$\frac{1}{2}t = 3.3$ [subtracting 3]

$\therefore\ t = 6.6$ hours, so the tank fills in 6.6 hours.

Note: The gradient of the straight line is the rate at which the tank fills. In many similar situations the gradient is the **rate of change**.

Exercise 10e

1. An elastic band initially of length 20cm is stretched with a steadily increasing force. Each increase in force of 1 newton causes an increase in length of 0.125cm.

   a. Copy and fill in this table:

Force applied in newtons (F)	0	1	2	3	4
Length of band in cm (L)					

   b. Draw a graph of length against force and find the equation relating the length of the band to the stretching force.
   c. The band will snap when it reaches 27.6cm long. Find the force that will cause the band to snap.

2. A tank contains 10 litres of water. A small hole is drilled in it and water escapes at a constant rate of 0.12 litres per minute.
   a. Draw a graph of volume of water, v, against time, t.
   b. Find the equation relating volume to time.
   c. At what time is the volume 5.2 L?

3. A pupil reads 0.8 pages of a book per minute. At the start of a reading lesson she starts at the top of page 13.
   a. Draw a graph of how much she has read, P, against time, t.
   b. Find the equation relating how many pages she has read with the time she has been reading during the lesson.
   c. What is the maximum number of pages the book can have if she finishes it in a 1 hour reading lesson?

4. A company car whose initial purchase price is $20 000 is depreciated in value at the rate of 25¢ a kilometre.
   a. Find the equation relating value of the car, v, to distance, d, it has travelled.
   b. How far will the car have to travel before it has a value of $15 000?

5. A tank is filled at a constant rate. 10 minutes after filling is started, the tank contains 4.8 litres of water. After 35 minutes the tank contains 7.3 litres of water.
   a. Find the rate at which the tank is being filled.
   b. Find the initial volume of fluid in the tank.
   c. Find how long it takes to fill if the tank has a maximum capacity of 60 L.

6. A car is 30 km from a town 10 minutes after the start of its journey. 25 minutes later it is 65.42 km from the town.
   a. Find the equation relating distance, in kilometres, d, from the town to time travelled in hours, t.
   b. How close was the car to the town when it began its journey?

Problems and Investigations

1. A student with careless habits was given the task of recording weights of some specimens on different days. It is well known that the weight increases at a constant rate with time. The student forgets his scales on 2 of the days and guesses the weights. Here are the results which the student hands in:

weight (g)	0.55	0.68	0.95	1.05	1.35	1.55	1.75
day	1	2	3	4	5	6	7

Which were the days he forgot his scales?

2. The same student from the previous problem correctly weighed another specimen on 5 different days. He recorded the weights but not the days he weighed them on. He remembers that he did the last one yesterday and the previous weighing 4 days before that. The specimen increases its weight at a steady rate. The weights he has recorded are 0.77, 1.25, 1.49, 1.85, 2.33. Yesterday was December 30. What were the dates of his previous readings?

11. PARALLEL AND PERPENDICULAR LINES

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 GEOMETRY

- *to write and use equations of straight lines in 2 dimensions (including a point and the equation of a parallel or perpendicular line)*

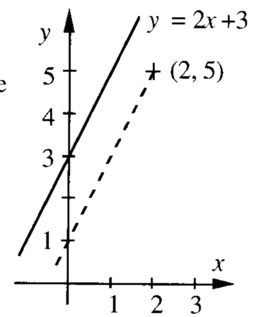
Parallel Lines

Two lines are *parallel* if they have the *same gradient.*

Example A: Find the equation of the line parallel to $y = 2x + 3$ which goes through (2, 5).

Solution:

i. Graphical: The line required, shown dotted, has gradient 2 because it is parallel to the line $y = 2x + 3$. From the graph, the y intercept of the line is 1 and the equation of the line is found by substituting m = 2 and c = 1 in $y = mx + c$ to give $y = 2x + 1$.



ii. Algebraic: The equation of the straight line is found by substituting m = 2 and $(x_1, y_1) = (2, 5)$ into $y - y_1 = m(x - x_1)$ giving:

$y - 5 = 2(x - 2)$

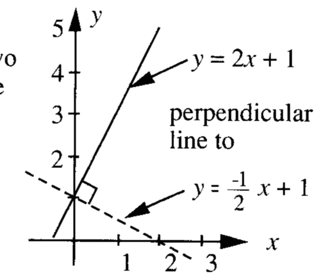
$\therefore\ y - 5 = 2x - 4$ [expanding]

$\therefore\ y = 2x + 1$ [adding 5]

Note: The graphical method is simplest but is limited to problems involving *integral* gradients and intercepts.

Perpendicular Lines

The following example *compares* the gradients of two perpendicular lines and shows that the product of the gradients of the two perpendicular lines is -1.



Example B: Draw the line $y = 2x + 1$ and the line perpendicular to it with the same y intercept. Find the gradient of the perpendicular line.

Solution: The perpendicular line (dotted) goes through (0, 1) and (2, 0).
Going from (0, 1) to (2, 0) the gradient of the perpendicular line is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 1}{2 - 0} = -\frac{1}{2}.$$

Note: (gradient of $y = 2x + 1$) × (gradient of perpendicular line)
is $2 \times \frac{-1}{2} = -1$

The above result is true for any two perpendicular lines and can be written:

$m_1 \times m_2 = -1$, where m_1 and m_2 are the gradients of two perpendicular lines.

Example C: Find the equation of the line perpendicular to $y = \frac{2}{3}x + 1$ going through (2, 1).

Solution: The gradient of the new line is $\frac{-3}{2}$ [since $\frac{2}{3} \times \frac{-3}{2} = -1$]

$\therefore\ y - 1 = \frac{-3}{2}(x - 2)$	[substituting into $y - y_1 = m(x - x_1)$]
$\therefore\ 2y - 2 = -3(x - 2)$	[multiply by 2]
$\therefore\ 2y - 2 = -3x + 6$	[expanding]
$\therefore\ 2y = -3x + 8$	[adding 2]
or $2y + 3x - 8 = 0$	[adding 3, subtracting 8]

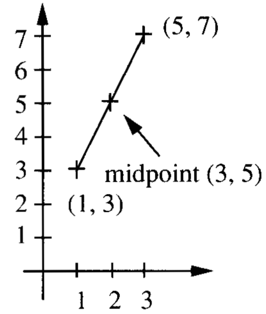
Note: The equation could be written in the form $y = \frac{-3}{2}x + 4$ [dividing by 2]

Exercise 11a

Find the equation of the line:
1. parallel to $y = 2x - 1$ going through (3, 4).
2. parallel to $y = \frac{x}{2} + 5$ going through (-1, 4).
3. parallel to $y = 6 - 3x$ going through (2, 1).
4. parallel to $y = 4 - \frac{2}{3}x$ going through (3, 4).
5. perpendicular to $y = 3x + 2$ going through (6, 4).
6. perpendicular to $y = 2x - 7$ going through (4, 5).
7. perpendicular to $y = \frac{-1}{2}x$ going through (1, -3).
8. perpendicular to $y = \frac{2}{3}x + 4$ going through (4, -3).
9. parallel to $2x - 3y + 11 = 0$ going through (2, -1).
10. parallel to $3x + y = 2y - 4x + 7$ going through (-2, -7).
11. perpendicular to $3y - 2x + 11 = 0$ going through (2, -2).
12. perpendicular to $4x + 3y = 15$ going through (1, -3).
13. perpendicular to $2x - 3y = 4x + 2y - 11$ going through (2, 10).
14. perpendicular to $4x - 2y = 5y + 7$ going through (-1, 0).
15. Find k if the lines $kx - 2y + 5 = 0$ and $3x + 4y + 1 = 0$ are:
    a. perpendicular b. parallel

Midpoint of a Line Segment

The **midpoint** of a **line segment** is the point halfway between the end-points.



Example D: The diagram shows that the midpoint of the line segment joining (1, 3) and (5, 7) is (3, 5).

Note: The x co-ordinate is $\frac{1+5}{2} = 3$.

The y co-ordinate is $\frac{3+7}{2} = 5$.

The result in the example D can be generalised:

The co-ordinates of the midpoint of the line segment joining (x_1, y_1) and (x_2, y_2) are: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Example E: The co-ordinates of the midpoint of the line segment joining (-1, 5) and (2, 7) are $\left(\frac{-1+2}{2}, \frac{5+7}{2}\right) = (\frac{1}{2}, 6)$

Exercise 11b

1. What is the midpoint of the line segment joining:
   a. (5, 8), (2, 4) b. (-3, -4), (-1, 8) c. (5, -7), (3, 1)
   d. (2, 1), (1, -1) e. (-1, 5), (2, 3)

2. Find the equation of the line parallel to $y = 2x + 3$ going through the midpoint of the line segment joining (2, 3), (4, 5).

3. Find the equation of the line parallel to $y = x + 11$ going through the midpoint of the line segment joining (-2, -1), (6, 9).

4. Find the equation of the line parallel to $y = \frac{(x+3)}{2}$ going through the midpoint of the line segment joining $(\frac{-1}{2}, 3)$, $(2\frac{1}{2}, -1)$.

5. Find the equation of the line perpendicular to the line segment joining (-4, 5), (-2, 1) and going through its midpoint.

6. Find the equation of the line perpendicular to the line segment joining (2, 0), (-4, 6) and going through its midpoint.

7. Find the equation of the line making an angle of 63.43° with the positive horizontal direction going through the midpoint of the line segment joining (-4, -2) and (6, 4).

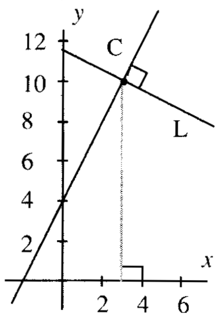
8. Find the equation of the line making an angle of 18.43° with the positive horizontal direction going through the midpoint of the line segment joining (-8, 1) and (2, 3).

9. Find the equation of the line making an angle of 146.3° with the positive horizontal direction going through the midpoint of the line segment joining (0, 0) and (2, 4).

Problems and Investigations

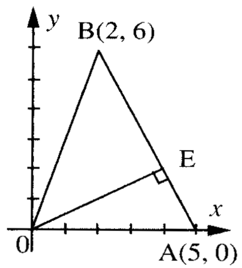
1. Find the equation of L, the line which meets $y = 2x + 4$ at 90° at the point C.

   [**Hint:** in these three problems it is very helpful to draw diagrams.]



2. In the diagram, the co-ordinates of A and B are (5, 0) and (2, 6) respectively. OE is perpendicular to AB. Find the co-ordinates of E.

3. Investigate the claim that the area of the triangle ABC with A (1, 2), B (4, 6), C (-4, 10) is greater than the area of the triangle ADE with D (4, -5), E (-5, -8).



12. THE INTERSECTION OF TWO STRAIGHT LINES

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 GEOMETRY

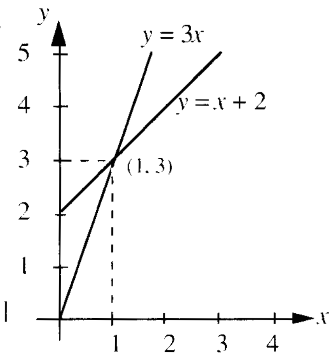
- *to find the co-ordinates of the intersection of a straight line and a curve, given their equations*

The point where two straight lines meet can be found *graphically* or *algebraically*. An algebraic solution involves solving two **simultaneous equations**.

Example A: Find the point of intersection of the lines $y = x + 2$ and $y = 3x$.

Solution:

i. Graphical: The two lines $y = 3x$ and $y = x + 2$ are drawn and the point of intersection (1, 3) is read off.



ii. Algebraic: The equations of the two lines are solved simultaneously by elimination.

$y = 3x$ .....(1)

$y = x + 2$ .....(2)

$0 = 2x - 2$ [subtracting (2) from (1)]

$2x = 2$ [rearranging]

$\therefore\ x = 1$

$\therefore\ y = 3$ [substituting $x = 1$ into the equation of either line]

$\therefore$ The point of intersection of $y = 3x$ and $y = x + 2$ is (1, 3).

Example B: Two women, Jill and Judy, weigh 78kg and 80kg. They go on a diet, starting on the same day. Jill loses 0.23kg a day and Judy loses 0.35kg a day. How much time will pass before they weigh the same?

Solution: Let w represent weight and t the number of days the diet has lasted. After t days Jill will have lost 0.23t and her weight is w = 78 – 0.23t. Similarly, Judy's weight is w = 80 – 0.35t.

Solving simultaneously, we get:

$$\begin{aligned} w &= 80 - 0.35t \\ w &= 78 - 0.23t \quad \text{[subtracting]} \\ \hline 0 &= 2 - 0.12t \\ \therefore 2 &= 0.12t \quad \text{[re-arranging]} \\ \therefore t &= 16\tfrac{2}{3} \end{aligned}$$

Hence their weights will be the same on the seventeenth day of the diet.

Use of Calculators

The solution of simultaneous equations is enormously simplified by the use of graphical calculators. The graphs of the straight line are drawn and the point of intersection read off.

As mentioned in Chapter 7, simultaneous equations, there are even more direct ways of solution on many calculators.

Concurrence

Two or more lines are **concurrent** if they all pass through the same point.

Example C: Show that $y = 3x$, $y = 2x + 1$ and $y = x + 2$ are all concurrent.

Solution: From example A, $y = 3x$ and $y = x + 2$ pass through (1, 3). Substituting (1, 3) into $y = 2x + 1$ gives $3 = 2 \times 1 + 1$, which is true. Hence (1, 3) lies on $y = 2x + 1$ also. Thus all three lines pass through (1, 3).

Exercise 12

1. Find the point of intersection of the following pairs of straight lines:
   a. $y = 2x,\ y = 3x - 1$ b. $y = x + 3,\ y = 2x + 3$
   c. $y = 3,\ y = x + 1$ d. $x = 2,\ y = x - 1$
   e. $y = \frac{1}{2}x + 1,\ y = 2x + 4$ f. $y = x,\ 2x + 3y = 5$
   g. $2x - 3y = 1,\ y = x - 1$ h. $3x + 2y = 11,\ 2x - y = 5$
   i. $2x + 5y = 11,\ 3y = 4x - 9$ j. $3x + 2y - 7 = 0,\ 5x - 6y - 12 = 0$

2. Show that the following sets of straight lines are concurrent and find their point of concurrence:
   a. $x = 1, y = 4,\ y = x + 3$ b. $y = 2x + 3,\ y = 3x + 2,\ y = 5x$
   c. $y = \frac{x}{2} + 1,\ y = x,\ 3y = x + 4$

3. Find k if the following sets of lines are concurrent:
   $y = x - 1,\ y = 2x + 1,\ y = kx + 2$

4. Prove that the following sets of lines are not concurrent:
   a. $4y - 3x - 6 = 0, 4y + 3x + 6 = 0,\ x = 2$
   b. $y = x + 1,\ y = 2x + 3,\ y = 3$
   c. $y = x, y = x + 3, y = 2x - 1$

5. Prove $2x - 3y = 5$
   and $4x = 6y - 7$ are lines which never intersect.

Problems and Investigations

1. Refer to example B and find out how long it would take Jill and Judy to reach the same weight if Jill's initial weight was 82kg, Judy's was 84kg, Jill lost 0.35kg per day and Judy lost 0.4kg per day.

2. David saves on average \$36 per week. He started on January 1 with \$50 in his account. William saves nothing. He started with \$1 000 in his account on January 1 and spends on average \$24 per week from this account. How many weeks will pass before David has more than William?

3. Find the length of the perimeter of the triangle formed by the following straight lines: $y = x$, $y = 2x + 3$, $y = 3x + 1$

4. a. Prove that the triangle formed by the lines $y = x + 3$, $y = 3 - x$ and $y = 2x + 6$ is right-angled.
   b. Find the area of this triangle.

5. Investigate the claim that $x - 2y + 3 = 0$, $3x - 4y + 7 = 0$ and $t(x - 2y + 3) - u(3x - 4y + 7) = 0$ are always concurrent, where t and u are constants.

13. GRAPHS II

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 ALGEBRA

- *to model a variety of situations using graphs*
- *to sketch graphs and investigate the graph of a function, using a calculator and plotting points if necessary*
- *to use graphical methods to investigate a pattern in data and, where appropriate, identify its algebraic form*
- *to describe the relationship between members of families of curves in terms of transformations*

Introduction

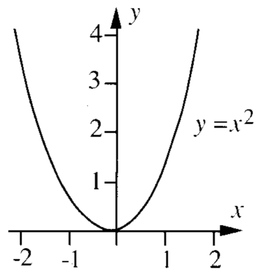
This chapter is concerned with **sketching** graphs rather than plotting them as was the case in Graphs I. A sketch includes details such as x and y intercepts, turning points and axes of symmetry.

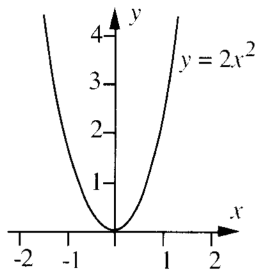
Parabolas

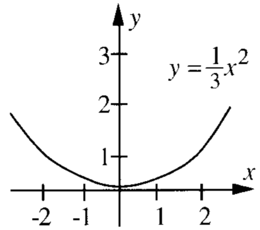
Parabolas are the graphs of **quadratic functions**. Quadratic functions are those of the type $y = Ax^2 + Bx + C$ where A, B and C are constant numbers.

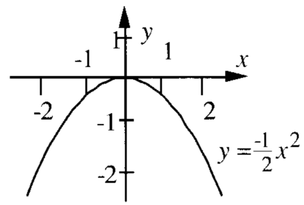
When B, C are both zero, the graphs are of the type $y = Ax^2$.

Example A: Sketches of $y = x^2$, $y = 2x^2$, $y = \frac{1}{3}x^2$ and $y = \frac{-1}{2}x^2$ are drawn by plotting a few points and joining them.









Note: The graph of $y = 2x^2$ is the graph of $y = x^2$ *compressed horizontally*.

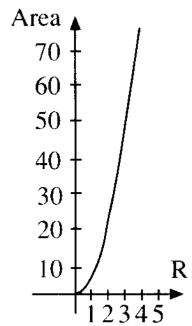
The graph of $y = \frac{1}{3}x^2$ is the graph of $y = x^2$ *stretched horizontally*.

The graph of $y = \frac{-1}{2}x^2$ is the graph of $y = x^2$ *stretched horizontally* and *inverted*.

Example B: Draw the graph for the area of a circle when graphed against radius.

Solution: This table gives some radii and the corresponding area. [Area = πR^2 when R is radius.]

Radius (R)	0	1	2	3	4	5
Area (πR^2)	0	3.1	12.6	28.3	50.3	78.5



The graph is 'half a parabola' of the type from example A. We only get half a parabola as it is not possible to have negative radii.

Quadratic functions of the type $y = A(x - B)^2 + C$ are sketched by finding the **intercepts**, **turning points** and **maximum** and **minimum values** as shown in the following example.

Example C: Note that for the graph of $y = 2(x - 1)^2 + 3$, the minimum value of y occurs when $x = 1$. Substituting $x = 1$ in $y = 2(x - 1)^2 + 3$ gives a minimum value for y of 3.

Note that (1, 3) is a turning point.

The y intercept occurs when $x = 0$

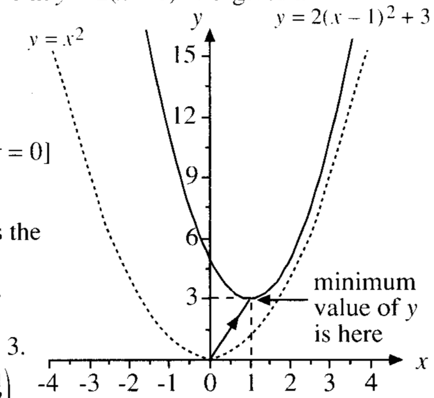
$\therefore\ y = 2(0 - 1)^2 + 3$ [substituting $x = 0$]

$y = 5$

Note: The graph of $y = 2(x - 1)^2 + 3$ is the graph of $y = x^2$ which has been:

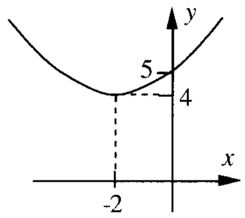
a. compressed by a factor of 2.
b. shifted to the right by 1.
c. shifted vertically upward by 3.

ie translated by the vector $\begin{pmatrix}1\\3\end{pmatrix}$.



The method of sketching illustrated in Example B is important because all quadratic functions can be expressed in the form $y = A(x - B)^2 + C$.

Example D: Find the equation of the parabola shown in the diagram. Write the equation in the form $y = A(x - B)^2 + C$.



Solution: The vertex is (-2, 4) so the equation is:

$y = A(x + 2)^2 + 4$.

The parabola goes through (0, 5) thus:

$\therefore\ 5 = A(0 + 2)^2 + 4$ [substituting $x = 0$, $y = 5$ into $y = A(x + 2)^2 + 4$]

$\therefore\ 5 = 4A + 4$

$\therefore\ A = \frac{1}{4}$ [solving the equation]

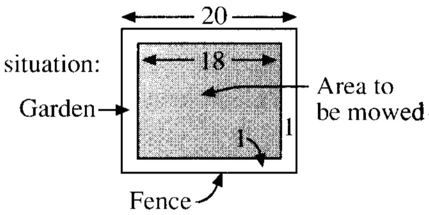
hence the equation is $y = \frac{1}{4}(x + 2)^2 + 4$.

Example E: A gardener has to mow a number of square lawns. The time it takes to mow a lawn involves a fixed time, independent of the size of the lawn where the lawnmower is taken off the truck, etc. This time is 5 minutes. Each square metre takes 1.5 seconds to mow. The gardener does not mow within one metre of the fence as there are gardens there.

a. Find the time it takes to mow a lawn whose fences on each side measure:
   i. 20m ii. 45m iii. 60m
b. Find the equation which represents the time it takes to mow lawns of this type.
c. Draw a sketch of a graph of time to mow against length of fence.

Solution:

a. i. This diagram illustrates the situation:



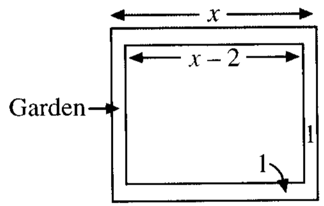
Area to be mowed is $(20 - 2)^2$.

$\therefore$ time is $\frac{1.5\,(20-2)^2}{60} + 5$ minutes [$1.5\,(20 - 2)^2$ gives time in seconds, dividing by 60 converts into minutes.]

$= 13.1$ minutes

Similarly the time taken to mow a lawn with fences of length 40m is 41.1 minutes and to mow a lawn with fences of length 60m is 89.1 minutes.

b. Using similar reasoning to that in a. for a garden whose side fence is of length x we get the diagram below:

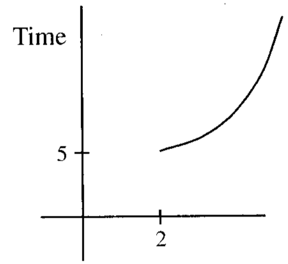


The time spent using the mower is: $1.5(x - 2)^2$ [in seconds]

$= \frac{1.5\,(x-2)^2}{60}$ minutes

Total time is $\frac{1.5\,(x-2)^2}{60} + 5$ minutes [adding constant of 5 minutes to include time taken for fixed tasks]

c.



The graph is half a parabola as it is not possible to have a lawn whose fence length would be less than 2. [Why not?]

Quadratic functions which are factorised or can be rearranged so that they factorise are easily sketched.

Example F: $y = x^2 - 6x + 5$ can be factorised to give $y = (x - 1)(x - 5)$.

The y intercept is found by making $x = 0$:

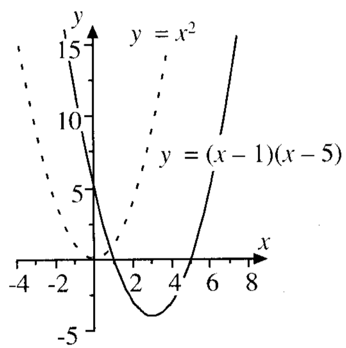
$\therefore\ y = 5$

The x intercepts are found by making $y = 0$:

$\therefore\ (x - 1)(x - 5) = 0$

$\therefore\ x = 1$ or $x = 5$

The axis of symmetry lies midway between the x intercepts and hence is the vertical line through $x = 3$.



The turning point will lie on the axis of symmetry and will have the vertical co-ordinate: $y = (3 - 1)(3 - 5) = -4$ [substituting $x = 3$ into $y = (x - 1)(x - 5)$]

$\therefore$ the turning point is (3, -4)

Note: $y = x^2 - 6x + 5$ can be written as $y = (x - 3)^2 - 4$ which is in the form $y = A(x - B)^2 + C$ and thus could be sketched in the way described in example C.

Any quadratic function in the form $y = Ax^2 + Bx + C$ can be written in the form $y = A(x - D)^2 + E$, which is then easily sketched using the method in example C.

Use of Graphical Calculators

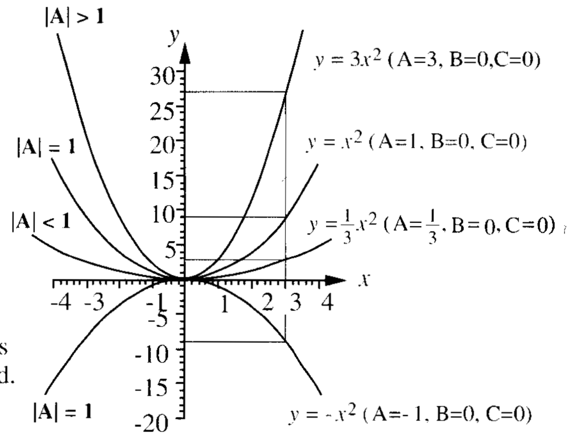
Students using such calculators will find the task of sketching graphs enormously simplified. Once drawn, the use of the TRACE function will enable easy location of such features as maximum and minimum points, intercepts, etc.

Summary

The relationship of various parabolas, written in the form $y = A(x + B)^2 + C$, to the parabola $y = x^2$ is summarised below.

a. The effect of A:
A alters the vertical co-ordinates of $y = x^2$ by the factor A.
If |A| > 1 the graph is *compressed* horizontally.
If |A| < 1 graph is *stretched* horizontally.

If A is *negative* the graph is *reflected* (ie appears 'upside down') in the x axis as well as being compressed or stretched.



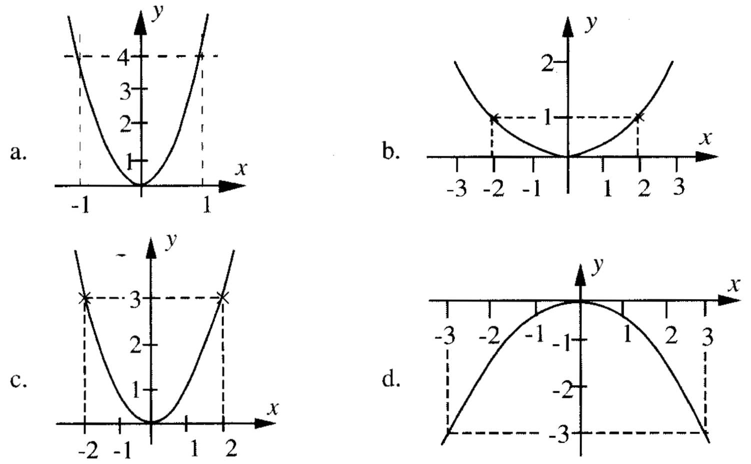
b. The effect of B:
If B is *positive*, the graph of $y = x^2$ is shifted B units to the *left*. [Example D]
If B is *negative*, the graph of $y = x^2$ is shifted B units to the *right*. [Example C]

c. The effect of C:
If C is *positive* the graph of $y = x^2$ is shifted |C| units *upward*. [Example C]
If C is *negative* the graph of $y = x^2$ is shifted |C| units *down*. [Example F]

Exercise 13a: Graphs

1. Sketch graphs of the following functions:
a. $y = 3x^2$ b. $y = \frac{1}{2}x^2$ c. $y = -x^2$ d. $y = \frac{2}{3}x^2$ e. $y = -4x^2$

2. Write down the equations of the parabolas shown in the graphs below.



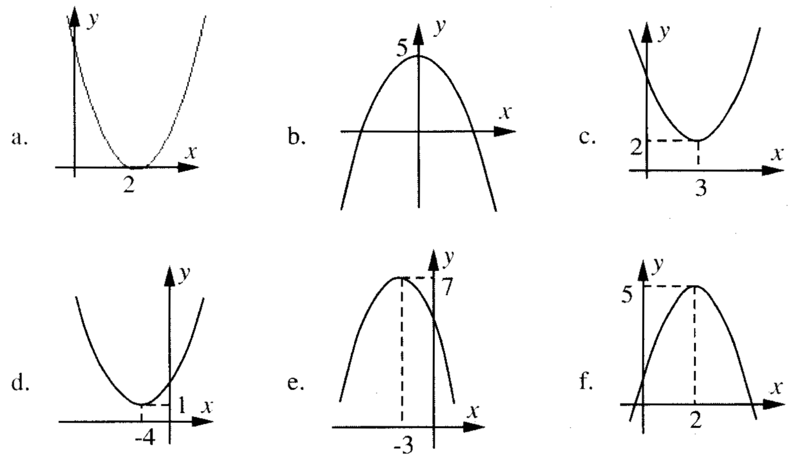
3. Draw the graph of the area of a semi-circle against radius.

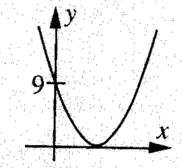
4. Draw the graph of the area of a circle against diameter.

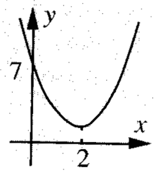
5. The cost of painting a wall is 30c a square metre. Draw the graph of cost in dollars of painting squares of various side lengths against side length.

6. By plotting the vertex and y intercept of the parabolas, sketch their graphs.
a. $y = (x - 1)^2 + 2$ b. $y = -(x + 2)^2 + 2$ c. $y = 3(x + 2)^2 - 4$
d. $y = -\frac{1}{2}(x + 2)^2 + 2$ e. $y = -2 - 2(x + 1)^2$

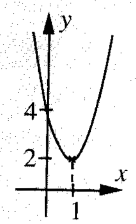
7. Write down the equations of these graphs which are all **congruent** [ie can be written in the form $y = \pm(x + A)^2 + B$] to $y = x^2$.

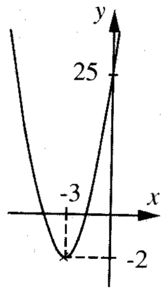


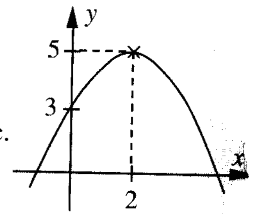
g.


h.


8. Write the equations of the following parabolas in the form $y = A(x - B)^2 + C$

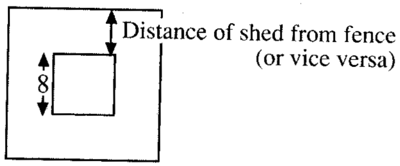
a.


b.


c.


9. A square shed of side length 5m is in the centre of a square field. Draw a graph of the area of grass in the field against length of side of the field.

10. A square shed of side length 8m is surrounded by a moveable square fence. The shed is in the middle of the area bounded by the fence. Draw a graph of the total area bounded by the fence against the distance of the fence from the shed. [See below.]



11. For each of the following make a tidy sketch of the graphs showing:
i. y intercept. ii. x intercepts. iii. turning points.
a. $y = (x + 1)(x - 3)$ b. $y = (x - 1)(x - 3)$ c. $y = (x + 2)(4 - x)$
d. $y = x^2 - 6x + 8$ e. $y = x^2 - 3x - 4$ f. $y = 2(x - 1)(x - 5)$
g. $y = \frac{1}{2}(x - 4)(x + 2)$ h. $y = 3x^2 - 6x - 9$

12. a. i. Find k if the parabola $y = k(x - 2)(x - 4)$ has a y intercept of (0, 16).
ii. Find the co-ordinates of the turning point.
b. Find B if the parabola $y = B(3 - x)(x + 5)$ has a turning point at (-1, 4).
c. Find L if $y = 25 - L(x - 2)^2$ has x intercepts of $\frac{-1}{2}$ and $4\frac{1}{2}$.

13. Find equations of each of the following parabolas:
a. x intercepts at (-3, 0), (1, 0) and y intercept at (0, -3).
b. x intercepts at 1 and 6, and y intercept at 4.
c. Turning point at (1, 5) and y intercept at (0, 3).
d. Turning point at (-1, 2) and y intercept at (0, 5).

Cubic Graphs

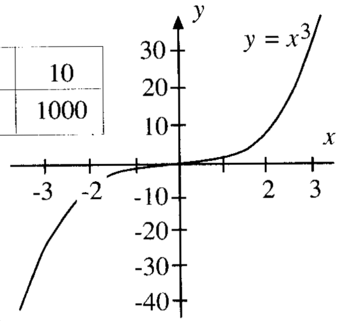
Cubic graphs are graphs of **cubic functions** which are polynomials of **degree 3**. They are of the form:

$$y = Ax^3 + Bx^2 + Cx + D$$

Cubic graphs are sketched by finding the x and y intercepts and sketching the curve through them. The section on calculus [chapter 21] shows how to find the co-ordinates of the turning points.

Example G: The graph of $y = x^3$ can be drawn by filling in a table then plotting the points:

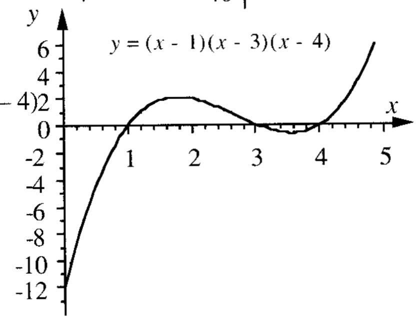
x	-10	-2	-1	0	1	2	10
x^3	-1000	-8	-1	0	1	8	1000



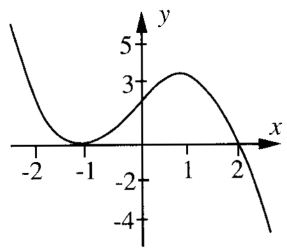
Example H: The graph of $y = (x - 1)(x - 3)(x - 4)$ can be sketched by finding the intercepts:
Substituting $x = 0$ in $y = (x - 1)(x - 3)(x - 4)$ gives the y intercept of -12.

The x intercepts are found by substituting $y = 0$ giving:

$(x - 1)(x - 3)(x - 4) = 0$
$\therefore x = 1, 3$ or 4

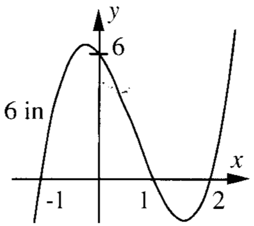


Example I: For the graph of $y = (x + 1)^2(2 - x)$:
the y intercept is 2. [substituting $x = 0$]
the x intercepts are found by putting $y = 0$ giving:
$(x + 1)^2(2 - x) = 0$, so the x intercepts are -1 and 2.



Note: The graph *touches* the x axis at -1.
In cubics like $y = (x + 1)^2(2 - x)$ where one of the factors of the polynomial is a perfect square, the graph will *touch* the x axis rather than cut it.

Example J: What is the equation of the graph shown?



Solution: The graph has x intercepts at -1, 1 and 2 so the equation of the graph is $y = A(x + 1)(x - 1)(x - 2)$. The graph has a y intercept of 6. Substituting $x = 0$, $y = 6$ in $y = A(x + 1)(x - 1)(x - 2)$ gives:

$6 = A(0 + 1)(0 - 1)(0 - 2)$ [substituting]

$\therefore 6 = 2A$

$\therefore A = 3$

Hence the equation of the graph is $y = 3(x + 1)(x - 1)(x - 2)$.

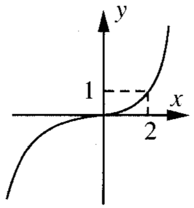
Exercise 13b

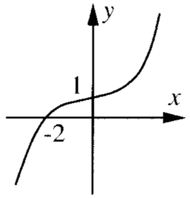
1. Sketch graphs of each of these functions:

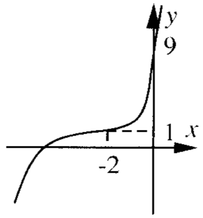
   a. $y = 2x^3$ b. $y = \frac{x^3}{3}$ c. $y = x^3 + 2$

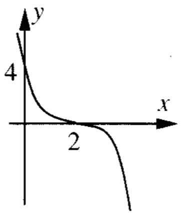
   d. $y = (x - 2)^3$ e. $y = \frac{1}{4}(x - 1)^3$ f. $y = 2 - (x + 1)^3$

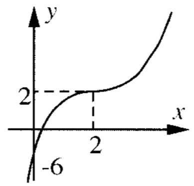
2. All of the following graphs involve some transformations of $y = x^3$ hence all can be written in the form $y = A(x - B)^3 + C$. Write down the equations of each of these graphs.

a.


b.


c.


d.


e.


3. Sketch the graphs of the following cubic functions:

   a. $y = (x - 1)(x - 2)(x + 4)$ b. $y = (x + 3)(x + 2)(x - 1)$

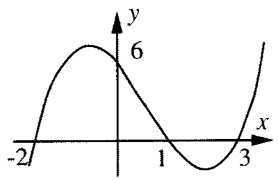
   c. $y = (2 - x)(2 + x)(x - 1)$ d. $y = x(x - 1)(x + 3)$

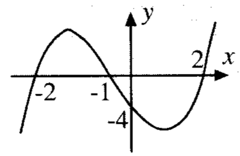
   e. $y = x(x - 1)^2$ f. $y = (x - 2)^2(x - 3)$

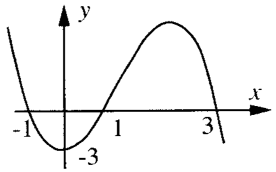
   g. $y = (x - 1)^2(2 - x)$ h. $y = -(2 - x)(x + 1)(x - 3)$

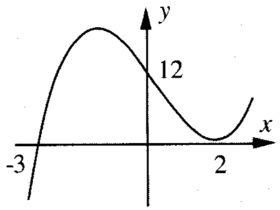
   i. $y = 2(x - 1)(x - 2)(x - 4)$ j. $y = \frac{1}{3}(x - 1)(x + 1)(x + 3)$

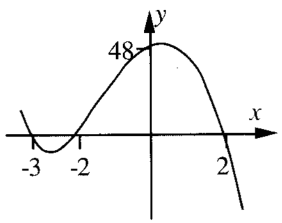
4. Write down the equations of the functions whose graphs are shown below: (they are all cubic)

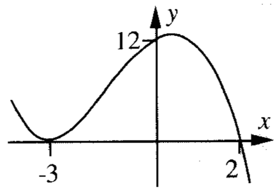
a.


b.


c.


d.


e.


f.


5. Draw the graph of the volume of a box against its length. Its height is two metres less than the length and its width is one metre less.

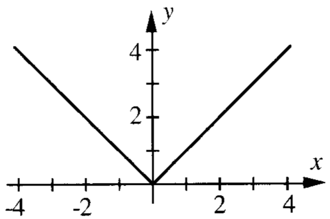
6. Draw a graph of the volume of a square based box against side length of the base. Its height exceeds the base length by one metre.

The Graph of the Modulus Function

The **modulus** or **absolute value** function makes all non-zero numbers positive. The modulus of x or the absolute value of x is written $|x|$.

Example K: To draw the graph of $y = |x|$ a table of values is set up and plotted:

x	-100	-3	-1	0	1	3	100
y	100	3	1	0	1	3	100

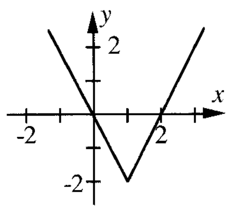


Note: The graph of $y = |x|$ is symmetrical about its minimum point.

Example L: Sketch the graph of $y = 2|x - 1| - 2$:

Solution: Evaluating different values for y yields the table below.

x	-2	-1	0	1	2	3	4
y	4	2	0	-2	0	2	4



After points are plotted and joined, the reader will note that the graph of $y = 2|x - 1| - 2$ is like the graph of $y = |x|$ but with:
a. translation to the right by 1 and down by 2 and
b. horizontal compression by a factor of 2.

This gives a minimum point of (1, -2) and a y intercept of 0.
The graph is symmetrical about the line $x = 1$.

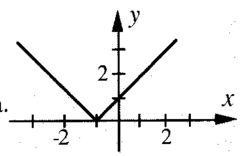
Note: Students with graphical calculators will find the modulus function called $y = \text{abs}(x)$. If the student has to draw the graph of $y = 3|x + 4|$ they would enter $y = 3 \text{ abs } (x + 4)$ into their calculator.

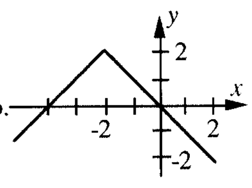
Exercise 13c

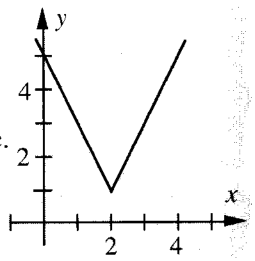
1. Sketch the graphs of each of the following:

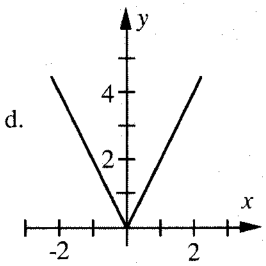
a. $y = \|x - 1\|$	b. $y = \|x + 3\|$	c. $y = \|x\| + 2$
d. $y = \| x\| - 3$	e. $y = \|x + 2\| + 1$	f. $y = \|x - 3\| + 2$
g. $y = \|x + 4\| - 3$	h. $y = - \|x\|$	i. $y = 3 - \|x \|$
j. $y = 2 - \|x + 1\|$	k. $y = -3 - \|x - 2\|$	l. $y = 2\|x\|$
m. $y = 2\|x - 1\| + 2$	n. $y = 3 \|x + 3\| - 1$	o. $y = 4 - 2\|x - 1\|$
p. $y = \|3 - x\|$	q. $y = 2 \|4 - x\| - 1$	r. $y = \|(x - 1)(x + 2)\|$
s. $y = \|x^2 - 1\|$		

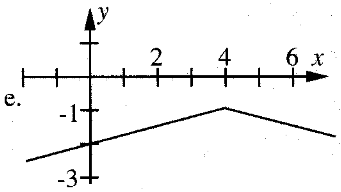
2. Write down the equations of the following graphs, each of which involves a transformation of $y = |x|$:

a. 

b. 

c. 

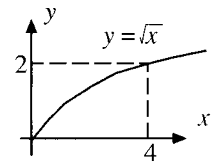
d. 

e. 

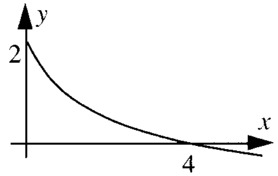
Graphs involving Square Roots

Example M: The graph of $y = \sqrt{x}$ is drawn from a table of values:

x	0	1	4	9	16
y	0	1	2	3	4



Example N: A sketch graph of $y = 2 - \sqrt{x}$ is the graph of $y = \sqrt{x}$ which has been reflected in the x axis and shifted up by 2.

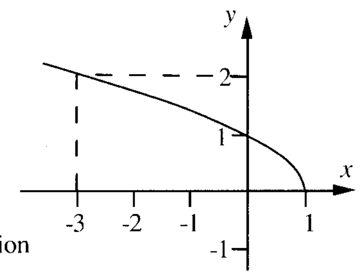


It arises from the table of values:

x	0	1	4	9	16
y	2	1	0	-1	-2

Example O: A sketch graph of $y = \sqrt{1 - x}$ is drawn from a table of values.

x	1	0	-3	-8
y	0	1	2	3



Note: The graph of $y = \sqrt{1 - x}$ is the reflection of the graph of $y = \sqrt{x - 1}$ in the line $x = 1$.

Exercise 13d

1. Sketch the graphs of each of the following:
   a. $y = \sqrt{x+3}$ b. $y = \sqrt{x-2}$ c. $y = \sqrt{x} + 1$ d. $y = \sqrt{x} - 2$
   e. $y = \sqrt{x-3} + 2$ f. $y = 3 - \sqrt{x}$ g. $y = 2\sqrt{x}$ h. $y = 3\sqrt{x+9} + 1$
   i. $y = 4 - 2\sqrt{x-3}$ j. $y = \sqrt{4-x}$ k. $y = \sqrt{9-x} + 2$
2. For each of the functions in question 1 give the domain and range.

Hyperbolas

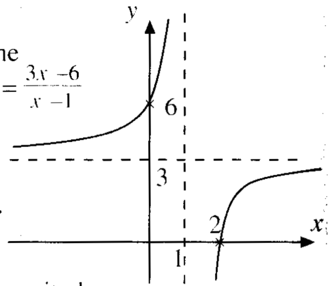
Hyperbolas are graphs of functions of the type $y = \frac{Ax+B}{Cx+D}$ where A, B, C and D are constants. Important features of hyperbolas are illustrated in the following examples.

Example P: The hyperbola $y = \frac{3x-6}{x-1}$ is drawn from a table of values:

x	-100	-10	-3	-2	-1	0	$\frac{1}{2}$	0.9	1	1.1	1.5	2	5	10	100
y	3.03	3.3	3.75	4	4.5	6	9	33	E	-27	-3	0	2.25	2.7	2.97

The graph which results is shown below.

Note:
a. There is no point on the graph with a y value of 3, ie, 3 is excluded from the **range**. The line $y = 3$ is called the **horizontal asymptote** of $y = \frac{3x-6}{x-1}$. As x gets larger and larger in magnitude, y gets closer and closer to 3.
b. There is no point on the graph with an x value of 1, ie, 1 is excluded from the **domain**. The line $x = 1$ is called the **vertical asymptote** of the hyperbola. As x gets closer and closer to 1, y gets greater and greater in magnitude.



Although it is always possible to sketch hyperbolas by plotting points, they are easily sketched by using the following procedure:

a. The *y intercept* is found by substituting $x = 0$.
b. The *x intercept* is found by substituting $y = 0$. This usually results in making the numerator of the function equal to zero and solving the resulting equation.
c. The *vertical asymptote* occurs at the restriction in the domain and is found by equating the denominator to zero and solving the resulting equation.

d. The *horizontal asymptote* is found by finding the limit of the function as the variable gets very large. The **cover up rule** can be used to find this limit:

Using the cover up rule, $y = \frac{Ax+B}{Cx+D}$ is written:

$y = \frac{Ax+\blacksquare}{Cx+\blacksquare}$ [covering up $\frac{+B}{+D}$]

$\therefore\ y = \frac{A}{C}$ [cancelling x's]

The vertical asymptote is thus $y = \frac{A}{C}$.

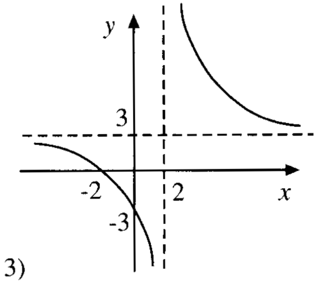
Example Q: To sketch $y = \frac{3x+6}{x-2}$, the x and y intercepts are found.

The y intercept is given by $y = \frac{-6}{2}$ [substituting $x = 0$ into $y = \frac{3x+6}{x-2}$]
$= -3$

The x intercept is given by $\frac{3x+6}{x-2} = 0$ [substituting $y = 0$ into $y = \frac{3x+6}{x-2}$]
$\therefore\ 3x + 6 = 0$ [multiplying by $x - 2$]
$\therefore\ x = -2$

The vertical asymptote is given by $x - 2 = 0$
$\therefore\ x = 2$

The horizontal asymptote, $y = 3$, is given by the cover up rule:
$\frac{3x+6}{x-2}$ becomes $\frac{3x}{x}$ which simplifies to 3.
This shows that as $x \to \infty$, $y \to 3$.



Note: The graph has point symmetry about (2, 3) which is the point of intersection of the two asymptotes.

Example R: The y intercept of $y = \frac{6}{3-x}$ is $y = \frac{6}{3}$ [putting $x = 0$]
$= 2$

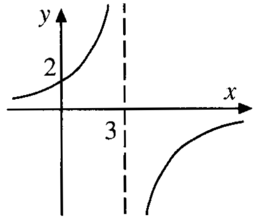
The x intercept is found by solving $\frac{6}{3-x} = 0$. [substituting $y = 0$]

Since there is no solution, there is no x intercept. Thus $y = 0$ is the horizontal asymptote.

The vertical asymptote is found by solving $3 - x = 0$; the asymptote is $x = 3$.

The cover up rule gives a horizontal asymptote $y = 0$.

A sketch of $y = \frac{6}{3-x}$ is shown.



Example S: A ship sinks and the survivors escape to an island where the local people give each person 5kg of food to last them until their rescuers arrive. In addition the survivors save 100kg of food from the ship which they divide equally among the survivors. The captain refuses any of the food brought ashore.

a. If x is the number of survivors, write an equation for the amount of food available to each member of the crew except the captain until the rescue ship arrives.

b. Draw a graph of this amount of food F against x.

Solution:

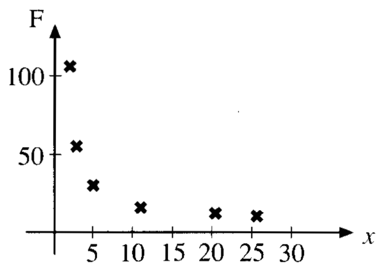
a. The amount of ship's food per member is $\frac{100}{x-1}$ [total food ÷ number of people]
[$x - 1$as we exclude the captain]

∴ total food available per member is $5 + \frac{100}{x-1}$ [add 5kg given by locals]

b. The graph below is obtained by plotting F (food available per member) against x. The table gives some values.

x	2	3	5	11	21	26	51
F	105	55	30	15	10	9	7

The graph is as below:



NB: i. The points are not joined as you can only have whole numbers for the numbers of survivors.

ii. The points all lie on a hyperbola. The reader might like to consider what would happen if $x = 1$.

Exercise 13e

1. Sketch the hyperbolas with the following characteristics:

	Vertical Asymptote	Horizontal Asymptote	x Intercept	y Intercept
a.	$x = 2$	$y = 2$	3	3
b.	$x = 3$	$y = 6$	1	2
c.	$x = 4$	$y = -3$	2	-1.5
d.	$x = 2$	$y = 3$	6	9
e.	$x = -1$	$y = 0$	None	4

2. Sketch the following:

a. $y = \frac{3x-6}{x-3}$ b. $y = \frac{4x-8}{x-4}$ c. $y = \frac{3x+6}{x-1}$ d. $y = \frac{x-3}{x-1}$

e. $y = \frac{2-x}{x+1}$ f. $y = \frac{2-4x}{x-2}$ g. $y = \frac{2x-5}{2x-1}$ h. $y = \frac{2}{x-1}$

i. $y = \frac{3}{2-x}$ j. $y = 1 + \frac{1}{x}$ k. $y = \frac{2}{x-1} - 1$ l. $y = 2 - \frac{3}{x+2}$

3. Write down the domain and range of each of the hyperbolas in Question 2.

4. Find the equation of the hyperbolas with the following details:

	Vertical Asymptote	Horizontal Asymptote	x Intercept	y Intercept
a.	$x = -3$	$y = 12$	-1	4
b.	$x = 0$	$y = 3$	2	none
c.	$x = 0$	$y = -1$	3	none
d.	$x = -2$	$y = 2$	-3	3
e.	$x = 5$	$y = -2$	-2	0.8

5. An amusement company runs games. On average per day 10 free games are played on each machine. The cost to the company per game consists of two parts: 5c of fixed costs, and a variable cost. The total cost to the company of a machine per day is $5.

a. Let x be the number of games which the customers pay for. What is the total number of games played per day?

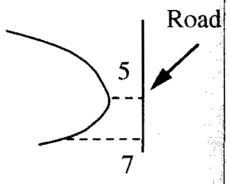
b. Let the variable cost be y cents per game. What is the total cost per day?

c. Write an equation relating y and x.

d. Draw the graph relating y and x.

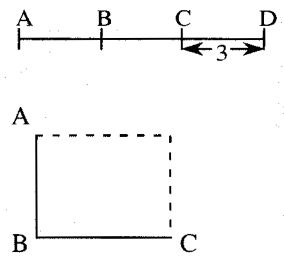
Problems and Investigations

1. The minimum distance of a traveller from a straight road is 5 km which she achieves after 2 hours travel. Initially her distance from the road is 7 km. The path she traces out as she travels is parabolic. Find a formula which gives her distance from the road in terms of time.



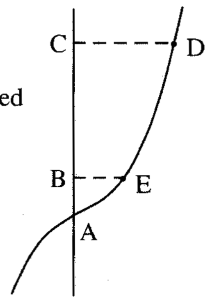
2. A wire AD is 8 metres in length. A length BD is removed from it to give AB. 3 metres are removed from BD to give BC [see diagram]. AB and BC are now joined at right angles to form 2 sides of a rectangle.
   a. Copy and complete this table:

BD(x)	AB	BC	Area of Rectangle
3	5	0	0
$3\frac{1}{2}$	$4\frac{1}{2}$	$\frac{1}{2}$	2.25
4	4	1	4
$4\frac{1}{2}$			
5			
6			
7			
8			

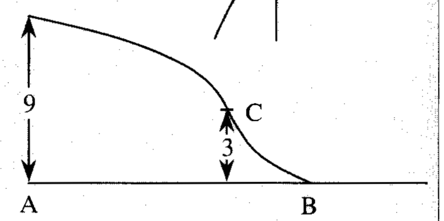


   b. Draw a graph of area against x.
   c. Write down a formula which relates the area of the rectangle to the length x of BD.
   d. What is the maximum area of the rectangle?

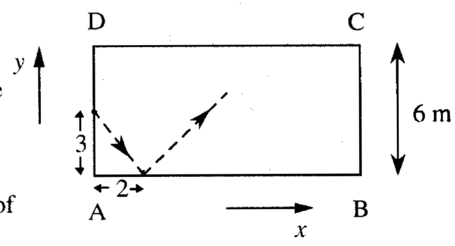
3. Two roads intersect. One is straight, the other is curved as shown in the form of a $y = K(x - L)^3 + M$ graph centred at A, where x is measured horizontally.
   The distance from A to B is 4 km. The distance from B to E is 2 km. The distance from C to D is 5 km. Find the distance from B to C.



4. A hill has a cross-section which is of the shape of $y = Kx^3$ cubic graph shown. The centre is at the point C shown with x measured vertically. Find the distance between A and B if K= 2.



5. A room is 6 m by 10 m. A ball is projected in a straight line from a position 3 m from corner A so as to hit a position 2 m from the same corner [see diagram].
   a. Taking corner A as an origin, write down the equation of the path of the ball from the time of projection until it hits CD.
   b. Write down the equation of the path of the ball from the time it first hits AB until the second time it hits AB.
   c. Where does it hit AB the second time?



6. Investigate the graphs of the following:
   a. $y = |x + 1| + |x - 2|$
   b. $y = 2|x + 3| - |x - 3|$
   c. $|x| + |y| = 4$
   d. $|x - 3| + |y + 2| = 5$
   e. $y^2 = (x - 1)(x - 2)(x - 4)$

14. EQUATION OF THE CIRCLE

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

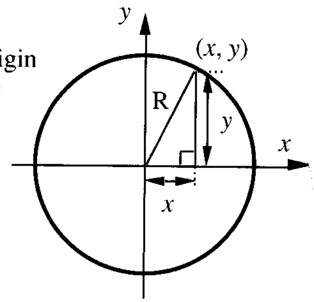
LEVEL 7 ALGEBRA

- *to sketch graphs*
- *to use graphing methods to investigate a pattern in data and, where appropriate, identify its algebraic form*

LEVEL 7 GEOMETRY

- *to find the co-ordinates of the intersection of a straight line and a curve given their equations*

Circles Centred at the Origin



To find the **equation of a circle** with centre at the origin (0, 0) and radius R, notice that any point (x, y) on the circle forms a right-angled triangle with the origin. This triangle has sides of length x, y and R.

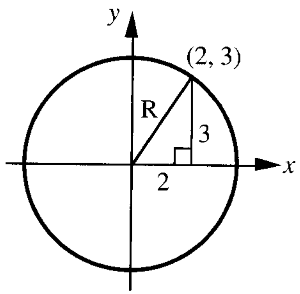
Using **Pythagoras' theorem**, the general equation of a circle, centred on the origin with radius R, is:

$$x^2 + y^2 = R^2$$

Example A: The radius of a circle, centred at the origin and passing through (2, 3) is R, where: $R^2 = 2^2 + 3^2$ [using Pythagoras' theorem]

$= 13$

$\therefore$ equation of the circle is $x^2 + y^2 = 13$



Where the circle cuts the x axis, $y = 0$

$\therefore x^2 = 13$

$\therefore x = \pm\sqrt{13}$

$\therefore$ co-ordinates of the x intercepts are $(\sqrt{13}, 0)$, $(-\sqrt{13}, 0)$.

Similarly the co-ordinates of the y intercepts are $(0, \sqrt{13})$, $(0, -\sqrt{13})$.

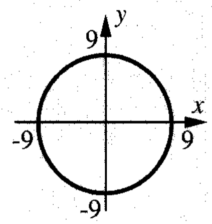
Graphical Calculators

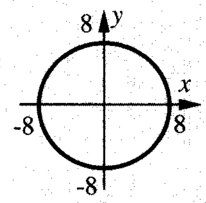
In order to draw graphs of the above type using a Casio graphical calculator it is necessary to draw the graph of $y = \sqrt{R^2 - x^2}$ then the graph of $y = -\sqrt{R^2 - x^2}$.

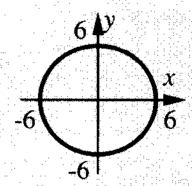
It is also possible to draw them using parametric methods however that is beyond the scope of this course although the reader owning a calculator such as the fx-7700 GB is encouraged to experiment with this facility.

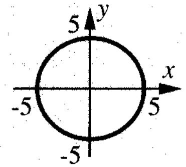
Exercise 14a

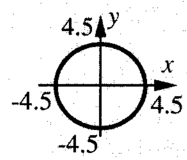
1. Write down the equations of each of the following circles:

a. 

b. 

c. 

d. 

e. 

2. What are the equations of the circles centred at the origin with radii:
   a. 6? b. 1? c. $\frac{1}{2}$? d. 0.3?

3. What is the radius of the circle which is centred at the origin and goes through:
   a. (2, 3)? b. (5, 4)? c. (-5, 2)? d. (0, -2)? e. (1, 2)?

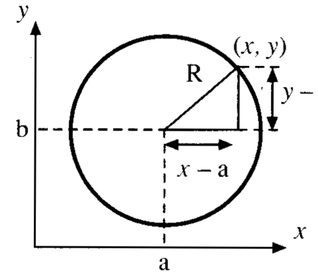
4. What are the equations of the circles centred at the origin going through:
   a. (-1, 2)? b. (3, 5)? c. (2, -1)? d. (3, -2)?
   e. (4, 6)? f. (5, -2)? g. (1.5, 2.5)? h. (-6, -4)?
   i. (-5, 0)? j. (0, 2)?

5. Which of the following sets of points all lie on circles centred at the origin. For those that do give the equation.
   a. $(\sqrt{2}, 0), (-1, 1), (1, 1), (1, -1)$
   b. $(3, 4), (4, -3), (5, 0), (0, -5)$
   c. $(2, 7), (3, 6), (5, 4)$
   d. $(-1, 0), (\frac{1}{2}, \frac{1}{2}), (0, 1), (-\frac{1}{2}, \frac{1}{2})$
   e. $(\sqrt{5}, 2), (2, -\sqrt{5}), (3, 0), (0, -3)$

Circle Centred at a Point (a, b)

The general equation of a circle with radius R and centre (a, b) can be found from the diagram in a similar way to that above. The equation is:

$$(x - a)^2 + (y - b)^2 = R^2$$



Example B: a. The equation of a circle centred at (1, 2) with radius 4 is found by substituting a = 1, b = 2 and R = 4 into $(x - a)^2 + (y - b)^2 = R^2$ giving:

$(x - 1)^2 + (y - 2)^2 = 4^2$
$\therefore\ (x - 1)^2 + (y - 2)^2 = 16$
$\therefore\ x^2 + y^2 - 2x - 4y + 1 + 4 = 16$ [expanding]
$\therefore\ x^2 + y^2 - 2x - 4y - 11 = 0$ [simplifying]

b. The equation of a circle centred at (-3, 4) with radius 2 is:

$(x - -3)^2 + (y - 4)^2 = 2^2$
$(x + 3)^2 + (y - 4)^2 = 4$
$x^2 + y^2 + 6x - 8y + 21 = 0$

Exercise 14b

Find the equations of the circles with the following features:

1. centre (2, 3), radius 5.
2. centre (-3, 1), radius 4.
3. centre (-2, -3), radius 2.
4. centre (1, 4), radius 3.
5. passing through (-3, 3) and centred at (1, 2).
6. passing through (2, 5) and centred at (4, 2).
7. diameter with end points (-3, 2), (4, 1).
8. diameter with end points (-1, -2), (3, 2).

Find the co-ordinates of the centre and the radius of each of the following circles:

9. $(x + 1)^2 + (y - 3)^2 = 16$
10. $(x - 2)^2 + (y - 4)^2 = 25$
11. $(x - 4)^2 + (y + 1)^2 = 36$

Find the equations on which the circles of the following sets of points lie:

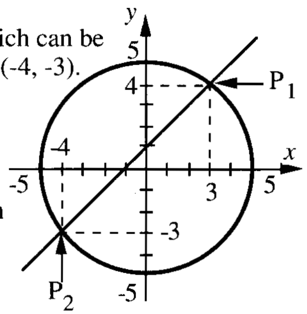
12. (4, 6), (4, -2), (-3, 5), (5, -1)
13. (9, 15), (13, 7), (3, -3), (11, 1)
14. (0, 14), (7, -3), (8, 2), (-5, -11)
15. (1, 2), (3, 2), (1, 4), (3, 4)
16. An outpost is surrounded by a circular fence 500m from the outpost. Any person coming into contact with it causes an alarm to ring. Ian was 230m east and 340m north of the outpost, Dave was 330m west and 215m south, Shane was 480m west and 140m south. Which one set off the alarm?

Intersection of a Circle with a Straight Line

A simple **graphical method** can be used where a graph is drawn and points of intersection read off. More accurate is an **algebraic method** where the equations of the circle and line are solved **simultaneously**. (Using a graphical calculator great accuracy can be obtained.)

Example D: Find the co-ordinates for the point of intersection of the line $y = x + 1$ with the circle $x^2 + y^2 = 25$.

Solution: Graphical
There are two points of intersection (P_1 and P_2) which can be read off from the above graph, $P_1 = (3, 4)$ and $P_2 = (-4, -3)$.



Solution: Algebraic
The equations $y = x + 1$ and $x^2 + y^2 = 25$ are solved simultaneously.

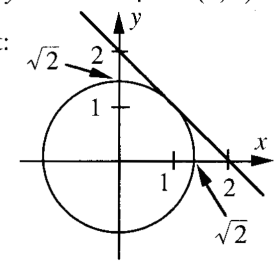
$x^2 + (x + 1)^2 = 25$ [substituting $y = x + 1$ in $x^2 + y^2 = 25$]
$\therefore\ x^2 + x^2 + 2x + 1 = 25$ [expanding]
$\therefore\ 2x^2 + 2x - 24 = 0$ [subtracting 25 and simplifying]
$\therefore\ x^2 + x - 12 = 0$ [dividing by 2]
$(x + 4)(x - 3) = 0$ [factorising]
$x = 3$ or -4

Substituting $x = 3$ and -4 into $y = x + 1$ gives the points of intersection (3, 4) and (-4, -3).

When the substitution method gives only one root to the quadratic equation, the line is a **tangent** to the circle.

Example E: Prove that $y = 2 - x$ is a tangent to $x^2 + y^2 = 2$ at the point (1, 1).

Solution: Substituting for y and solving gives 1 root:

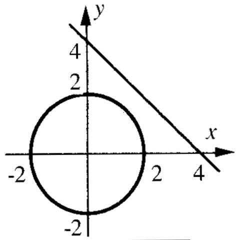


$x^2 + (2 - x)^2 = 2$ [substituting]
$\therefore\ x^2 + 4 - 4x + x^2 = 2$ [expanding]
$\therefore\ 2x^2 - 4x + 2 = 0$ [simplifying]
$\therefore\ x^2 - 2x + 1 = 0$ [dividing by 2]
$\therefore\ (x - 1)^2 = 0$ [factorising]
$\therefore\ x = 1$, confirming *one* root.

Then substitute $x = 1$ into $y = 2 - x$ to confirm that the point of intersection is (1, 1).

Where a straight line does not cross the circle, the resulting equation has *no real roots*.

Example F: Prove that $y = 4 - x$ does not cut the circle $x^2 + y^2 = 4$.



Solution:

Attempting to solve the equations simultaneously:

$x^2 + (4 - x)^2 = 4$ [substituting]

$\therefore\ x^2 + 16 - 8x + x^2 = 4$ [expanding]

$\therefore\ 2x^2 - 8x + 12 = 0$ [simplifying]

$\therefore\ x^2 - 4x + 6 = 0$ [dividing by 2]

$x = \dfrac{4 \pm \sqrt{-8}}{2}$ [substituting into $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$]

$x^2 - 4x + 6$ has no real roots as we cannot find the square root of a negative number (see the diagram). Therefore the line does not cut the circle.

Exercise 14c

1. Find the co-ordinates of the points of intersection, if any, of the circle $x^2 + y^2 = 4$ with the following straight lines:
   a. $y = 2$ b. $y = 1$ c. $x = 2$ d. $x = -1$
   e. $y = 1.5$ f. $y = 3$ g. $y = x$ h. $y = 2x$
   i. $y = 3x$ j. $x = 2y$

2. Find the points of intersection of the circle $x^2 + y^2 = 4$ with the straight lines:
   a. $y = x + 1$ b. $y = x - 1$ c. $y = 2x + 1$ d. $y = x + 2$
   e. $y = 2x - 2$

3. Find the co-ordinates of the points of intersection, if any, of the parabola $y = x^2$ with the following straight lines:
   a. $y = 4$ b. $x = 2$ c. $y = -9$ d. $y = x$
   e. $y = 2x$ f. $y = 3x$ g. $y = 2x + 3$ h. $y = 2x - 1$
   i. $y = 4 - 3x$ j. $y = x + 1$

4. Find the co-ordinates of the points of intersection, if any, of the hyperbola $y = \frac{1}{x}$ with the following straight lines:
   a. $y = 3$ b. $y = 5$ c. $x = 2$ d. $x = 4$
   e. $y = x$ f. $y = 2x$ g. $y = 2 - x$ h. $y = 2x + 1$
   i. $y = x - 1$ j. $x + y = 0$

Problems and Investigations

1. Each of the following lines is a tangent to the circle indicated.
   a. $3x + 4y = 25,\ x^2 + y^2 = 25$ b. $x + 2y = 5,\ x^2 + y^2 = 5$
   c. $2x - y = 5,\ x^2 + y^2 = 5$
   Investigate this claim.

2. Find K so that $x + y = K$ is a tangent to the circle $x^2 + y^2 = 9$.

3. Find the equations of the 2 tangents to the circle $x^2 + y^2 = 4$ from the point (0, 3).

4. Find the points of intersection of $y = x^2$ with the circle $x^2 + y^2 = 6$.

15. INDICES

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 ALGEBRA

- *to carry out appropriate manipulation and simplification of algebraic expressions (including fractional and negative exponents)*
- *to choose suitable strategies for finding solutions to equations and interpret the results*

Introduction

The important properties of indices are:

$x^m \times x^n = x^{m+n}$		$\frac{x^m}{x^n} = x^{m-n}$	$[x \neq 0]$
$(x^m)^n = x^{mn}$		$x^{-n} = \frac{1}{x^n}$	$[x \neq 0]$
$x^0 = 1$	$[x \neq 0]$	$(xy)^m = x^m y^m$	
$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$		$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$	
$\left(\frac{x}{y}\right)^{-m} = \left(\frac{y}{x}\right)^m$			

Negative Indices

Example A: Calculate the following:

a. 3^{-3} b. $\frac{1}{2^{-4}}$ c. $(-2)^{-2}$ d. $\frac{2^{-1}}{4}$

Solution: Calculations such as these are quite simple provided it is not necessary to give the answers as a fraction.

a. Using a calculator (*fx*-82)

3^{-3} is calculated by:

[3] [x^y] [3] [±] [=]

The answer is 0.037037. [to 6 decimal places]

b. $\frac{1}{2^{-4}}$ is calculated by:

[2] [x^y] [4] [±] [=] [$\frac{1}{x}$]

The answer is 16.

Similarly:

c. $(-2)^{-2} = 0.25$ [see p. 132] d. $\frac{2^{-1}}{4} = 0.125$

Example B: Calculate the following, giving the answers as fractions.

a. 2^{-3} b. $\frac{1}{3^{-2}}$ c. $\left(\frac{2}{5}\right)^3$ d. $\left(\frac{3}{7}\right)^{-2}$ e. $\frac{2^{-3}}{3^{-2}}$ f. $\frac{2^{-1}}{7}$

Solution: To carry out these calculations it is necesary to use the properties of indices.

a. $2^{-3} = \frac{1}{2^3}$ [since $x^{-m} = \frac{1}{x^m}$]

$= \frac{1}{8}$ [$2^3 = 8$]

b. $\frac{1}{3^{-2}} = 3^2$ [since $\frac{1}{x^{-m}} = x^m$]

$= 9$

c. $\left(\frac{2}{5}\right)^3 = \frac{2^3}{5^3}$ [as $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$]

$= \frac{8}{125}$

d. $\left(\frac{3}{7}\right)^{-2} = \left(\frac{7}{3}\right)^2$ [as $\left(\frac{x}{y}\right)^{-m} = \left(\frac{y}{x}\right)^m$]

$= \frac{7^2}{3^2}$ [as $\left(\frac{y}{x}\right)^m = \frac{y^m}{x^m}$]

$= \frac{49}{9}$

e. $\dfrac{2^{-3}}{3^{-2}} = \dfrac{3^2}{2^3}$ [as $\dfrac{x^{-n}}{y^{-m}} = \dfrac{y^m}{x^n}$]

$= \dfrac{9}{8}$

f. $\dfrac{2^{-1}}{7} = \dfrac{1}{7} \times 2^{-1}$ [as dividing by 7 is the same as multiplying by $\frac{1}{7}$]

$= \dfrac{1}{7} \times \dfrac{1}{2}$ [as $2^{-1} = \dfrac{1}{2^1} = \dfrac{1}{2}$]

$= \dfrac{1}{14}$ [multiplying fractions]

Exercise 15a

1. Calculate the following, giving the answers to no more than 4 decimal places:

a. 2^{-4} b. 1.2^{-5} c. 0.37^{-2} d. $0.37^{-1} + 2.3$

e. $(2.3)^{-2} + 4$ f. $\dfrac{1}{3^{-2}}$ g. $\dfrac{2}{(2.4)^{-3}}$ h. $(-3.4)^{-2}$

i. $\dfrac{1}{0.56^{-1}} + \dfrac{2}{0.5^{-2}}$ j. $(3.4^{-1} - (0.9)^{-2})^{-3}$ k. $(2.3)^{-3} + (-2.3)^{-2}$

2. Calculate the following, giving the answers as fractions:

a. 3^{-2} b. 4^{-3} c. 2^{-4} d. $\left(\frac{3}{4}\right)^3$

e. $\dfrac{2^3}{5^2}$ f. $\dfrac{1}{3^{-2}}$ g. $\dfrac{1}{4^{-2}}$ h. $\left(\frac{3}{5}\right)^{-2}$

i. $\left(\frac{4}{5}\right)^{-3}$ j. $\dfrac{2^{-2}}{3^{-3}}$ k. $\dfrac{3^{-3}}{5^{-2}}$ l. $\dfrac{3^{-1}}{8}$

m. $\dfrac{5^{-1}}{2}$ n. $7(5^{-1})$ o. 8×3^{-1} p. $2^{-1} \times 3^{-1}$

q. $3^{-1}\,5^{-1}$ r. $3^{-1}\,2^{-2}$ s. $\dfrac{5}{2^{-1}}$ t. $\dfrac{3}{2^{-2}}$

u. $\dfrac{3^{-2}}{4}$ v. $\dfrac{5^{-2}}{7}$ w. $3^{-1}\,6$ x. $3^{-2}\,6^2$

y. $2^{-1} \div 7$ z. $3^{-1} \div 4^{-1}$

3. Calculate the following, giving the answers as fractions:

a. $2^{-1} \times 3$ b. $3^{-2} \times 4^{-1}$ c. $(2^{-2} \times 3^{-1})^{-2}$ d. $(3^{-1}2^{-3})^2$

e. $2^{-1} + 3^{-1}$ f. $3^{-2} + 6^{-1}$ g. $2^{-1}\,3 + 5^{-1}\,3^2$ h. $2^{-2} - 8^{-1}$

i. $4^{-2} \div 8^{-1}$ j. $\dfrac{3^{-1}}{2^{-1}} + \dfrac{4^{-1}}{3^{-1}}$ k. $\dfrac{3^{-2}}{2^{-2}} - \dfrac{2^{-1}}{3^{-2}}$ l. $(3^1\,2^{-1} - 2^1\,3^{-1})^{-1}$

m. $(2^2\,3)^{-1} + 2^{-1}\,3^{-2}$ n. $5(2^3)^{-1} + 4^{-2}$ o. $7(2^{-1}) - 3(2^{-2})$

4. Write the following as fractions or integers with no powers:

a. 2×4^{-1} b. $2^{-1} \times 3^{-1}$ c. 4×2^{-1} d. $3^{-2} \times 5$

e. $\left(\frac{2}{3}\right)^{-1}$ f. $\left(1\frac{1}{2}\right)^{-1}$ g. $3\left(\frac{1}{3}\right)^{-1}$ h. $\frac{2}{3}\left(\frac{2}{3}\right)^{-2}$

i. $\dfrac{2^{-1}}{5} \times \left(\frac{1}{4}\right)^{-2}$

5. Write each of the following as the product of **primes** raised to powers:

e.g. $28 = 2^2 \times 7^1$, $\frac{3}{8} = 3 \times 2^{-3}$

a. 12 b. 200 c. 45 d. $\frac{25}{16}$

e. $\frac{2}{3}$ f. $\frac{4}{27}$ g. $\frac{16}{81}$ h. $\frac{64}{81}$

i. $2\frac{1}{4}$

Simplifying Algebraic Expressions

The solution to many of the following examples can be shortened or done in other ways.

Example C: The following are simplified and written with positive indices.

a. $x^{-5} = \dfrac{1}{x^5}$ [since $x^{-r} = \dfrac{1}{x^r}$]

b. $8a^{-2} = 8 \times a^{-2}$

$= 8 \times \dfrac{1}{a^2}$ [since $a^{-2} = \dfrac{1}{a^2}$]

$= \dfrac{8}{a^2}$

c. $\dfrac{2^{-1}a^{-3}}{5} = 2^{-1} \times a^{-3} \times \dfrac{1}{5}$ [dividing by 5 is the same as multiplying by $\frac{1}{5}$]

$= \dfrac{1}{2} \times \dfrac{1}{a^3} \times \dfrac{1}{5}$ [since $2^{-1} = \frac{1}{2}$ and $a^{-3} = \frac{1}{a^3}$]

$= \dfrac{1}{10a^3}$

Example D: Express $\dfrac{15x^{-3}y^2}{(3x^{-2}y^4)^2}$ without negative indices.

Solution:

$$\frac{15x^{-3}y^2}{(3x^{-2}y^4)^2} = \frac{15x^{-3}y^2}{3^2(x^{-2})^2(y^4)^2} \qquad [\text{since } (xy)^n = x^ny^n]$$

$$= \frac{15x^{-3}y^2}{9x^{-4}y^8} \qquad [\text{since } (x^n)^m = x^{nm}]$$

$$= \frac{5x^1y^{-6}}{3} \qquad [\frac{x^m}{x^n} = x^{m-n} \text{ and } \frac{15}{9} = \frac{5}{3}]$$

$$= \frac{5x}{3} \times y^{-6}$$

$$= \frac{5x}{3} \times \frac{1}{y^6} \qquad [\text{since } y^{-n} = \frac{1}{y^n}]$$

$$= \frac{5x}{3y^6}$$

Exercise 15b

1. Write each of the following without negative indices and as simply as possible: e.g. $3x^{-1} = \frac{3}{x}$

a. x^{-3}	b. y^{-2}	c. a^{-5}	d. b^{-4}	e. z^{-7}
f. y^{-6}	g. z^{-11}	h. $(xy)^{-1}$	i. $(abc)^{-1}$	j. $(xy)^{-2}$
k. $(2x)^{-1}$	l. $(2x)^{-2}$	m. $(3a)^{-2}$	n. $(2x)^{-3}$	o. $3x^{-1}$
p. ab^{-3}	q. $a^{-3}b^2$	r. $a^{-3}a^2$	s. $\frac{x^{-1}}{2}$	t. $\frac{x^{-3}}{4}$
u. $\frac{3x^{-2}}{5}$	v. $\frac{2a^{-3}}{7}$	w. $\frac{4a^{-2}}{9}$	x. $\frac{3x^{-2}}{5x^2}$	y. $\frac{4x^{-3}}{5x^2}$

2. Write the following using negative indices for the variable. Eg $\frac{3}{x^2} = 3x^{-2}$

a. $\frac{1}{x^6}$	b. $\frac{1}{y^5}$	c. $\frac{1}{z^2}$	d. $\frac{4}{x}$	e. $\frac{5}{y^2}$
f. $\frac{7}{z^5}$	g. $\frac{1}{2x^2}$	h. $\frac{1}{3a^4}$	i. $\frac{1}{5y^4}$	j. $\frac{2}{5x^2}$
k. $\frac{3}{4y^3}$	l. $\frac{3}{(2x)^2}$	m. $\frac{4}{(5y)^2}$	n. $\frac{x^2}{(2x)^3}$	o. $\frac{y^2}{(2x)^3}$

3. Write the following as simply as possible without using negative indices an as a single fraction:

a. $\frac{4x^{-2}}{8y^{-3}}$	b. $\frac{6a^{-3}}{9b^{-4}}$	c. $\frac{5x^{-2}}{25x^{-3}}$	d. $\frac{5^{-1}x^{-2}}{10^{-1}y^{-4}}$	e. $\frac{(2x)^2}{4x^{-3}y}$
f. $\frac{(4x^{-1})^2}{8x^{-1}y^{-3}}$	g. $\frac{(2x^{-2})^3y^{-2}}{x}$	h. $\frac{5^{-1}x^{-2}}{5x}$	i. $\frac{2^{-1}x^{-1}}{3y}$	j. $\frac{3^{-1}x^{-2}}{(3x)^2}$

k. $2^{-1}y^{-2}x$ l. $3^{-2}y^{-3}x^2$ m. $2(3^{-1}x^{-2})$ n. $4(5^{-1}x^{-3})y^2$
o. $x^{-1} + y^{-1}$ p. $x^{-1} + x^{-2}$ q. $x(2^{-1}y + y^{-1}x^{-1})$

Rational Powers

$x^{\frac{1}{2}}$ is $\sqrt{x}$, the **square root** of x. $x^{\frac{1}{3}}$ is $\sqrt[3]{x}$, the **cube root** of x.
Generally, the yth root of x can be written:

$$x^{\frac{1}{y}} = \sqrt[y]{x}$$

Proof: The following proof for $y = 3$ can be applied to any value of y.

$$\left[x^{\frac{1}{3}}\right]^3 = x^1 \qquad [\text{since } (x^m)^n = x^{mn}]$$

$$\therefore x^{\frac{1}{3}} = \sqrt[3]{x} \qquad [\text{taking cube roots}]$$

Generalising further:

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

Example E:

$$(-27)^{\frac{5}{3}} = \left[(-27)^{\frac{1}{3}}\right]^5 \qquad [\text{since } (x)^{\frac{m}{n}} = \left[(x)^{\frac{1}{n}}\right]^m]$$

$$= \left(\sqrt[3]{-27}\right)^5 \qquad [\text{since } (x)^{\frac{1}{n}} = \sqrt[n]{x}]$$

$$= (-3)^5$$

$$= -243$$

Example F: Write the following in index form, ie: a. $\sqrt{x}$ b. $\sqrt[5]{x^2}$ c. $\frac{1}{\sqrt[3]{x^2}}$

a. $\sqrt{x} = \sqrt[2]{x}$ [from the definition of $\sqrt{\ }$]

$= x^{\frac{1}{2}}$

b. $\sqrt[5]{x^2} = x^{\frac{2}{5}}$ [since $\sqrt[n]{x^m} = x^{\frac{m}{n}}$]

c. $\frac{1}{\sqrt[3]{x^2}} = \frac{1}{x^{\frac{2}{3}}}$ [since $\sqrt[n]{x^m} = x^{\frac{m}{n}}$]

$= x^{-\frac{2}{3}}$ [since $x^{-n} = \frac{1}{x^n}$]

Exercise 15c

1. Express the following in index form as simply as possible. e.g. $3\sqrt[4]{x^5} = 3x^{\frac{5}{4}}$

a. $\sqrt{a}$	b. $\sqrt{c}$	c. $\sqrt[3]{x}$	d. $\sqrt[5]{a}$	e. $\sqrt[6]{u}$
f. $\sqrt[5]{x^2}$	g. $\sqrt[7]{x^3}$	h. $\sqrt[8]{a^3}$	i. $\frac{1}{\sqrt{x}}$	j. $\frac{1}{\sqrt{y}}$

k. $\frac{1}{\sqrt[3]{a}}$ l. $\frac{1}{\sqrt[5]{x}}$ m. $\frac{1}{\sqrt[7]{x^2}}$ n. $\frac{1}{\sqrt[9]{x^5}}$ o. $\frac{1}{\sqrt[7]{a^3}}$

p. $\frac{1}{\sqrt[9]{a^8}}$ q. $4\sqrt[3]{a}$ r. $\frac{6}{\sqrt[3]{y}}$ s. $\frac{8}{\sqrt[3]{x^7}}$ t. $\frac{1}{7\sqrt[3]{x^2}}$

u. $\frac{2}{5\sqrt[4]{x^3}}$ v. $(\sqrt{x})^{-2}$ w. $\frac{\sqrt{x}}{\sqrt[3]{x}}$ x. $\frac{1}{\sqrt[3]{a}\,\sqrt[3]{a^4}}$ y. $\frac{\sqrt{x}}{\sqrt[3]{x}\,\sqrt[4]{x}}$

2. Express the following using radical (or root) signs. Eg $x^{\frac{2}{3}} = \sqrt[3]{x^2}$

a. $a^{\frac{3}{7}}$ b. $b^{\frac{3}{4}}$ c. $x^{\frac{2}{5}}$ d. $x^{\frac{-2}{3}}$ e. $y^{\frac{-3}{4}}$

3. Express the following in index form and as simply as possible:

a. $\sqrt{xy}$ b. $\sqrt{xy^2}$ c. $\sqrt{\frac{x^2}{y}}$

d. $\sqrt[3]{\frac{x^6}{y}}$ e. $\sqrt[5]{\frac{32x^5}{y^3}}$ f. $\sqrt[4]{x^{-5}y^4}$

g. $\sqrt{x} \times \sqrt[3]{a}$ h. $\frac{1}{\sqrt{a}} \times \sqrt{c}$ i. $\frac{1}{\sqrt[4]{b^{-2}}} \times \sqrt{b}$

Using Calculators to Evaluate $(-x)^n$

Calculators which have x^y and $(x)^{\frac{1}{y}}$ buttons can simplify calculations involving indices. Some calculators will not accept a negative base, so when attempting to calculate indices such as $(-3)^2$, the calculator will display E, showing that the calculator cannot calculate $(-3)^2$ directly. In such cases the following procedures are useful.

a. Calculation of $(-x)^n$ where x is positive and n is an integer.

> If n is *even*, $(-x)^n = (x)^n$.
> If n is *odd*, $(-x)^n = -(x)^n$.

Example G:

a. $(-4)^5 = -(4)^5$ [the index n is odd]
$= -1\,024$

b. $(-7)^{-2} = 7^{-2}$ [the index -2 is even]
$= 0.02$ [to 2 decimal places]

b. Calculation of $(-x)^{\frac{m}{n}}$ where m and n are integers and x is positive. The index $\frac{m}{n}$ should be simplified as much as possible. Then, regardless of whether $\frac{m}{n}$ is negative or positive:

If n is even, $(-x)^{\frac{m}{n}}$ cannot be calculated.

If n is odd and m is even, $(-x)^{\frac{m}{n}} = (x)^{\frac{m}{n}}$

If n and m are odd, $(-x)^{\frac{m}{n}} = -\left[(x)^{\frac{m}{n}}\right]$

Example H: $(-27.5)^{\frac{2}{3}} = (27.5)^{\frac{2}{3}}$ [since 2 is even and 3 is odd]
$= 9.11$ [using a calculator, to 2 decimal places]

Note: On a calculator $(-27.5)^{\frac{2}{3}}$ would be calculated:

2 7 . 5 x^y (2 ÷ 3) =

Exercise 15d

Use a calculator to work out each of the following:

1. 2^4
2. 3^{-2}
3. 0.23^{-3}
4. $(-2)^5$
5. $\left(\frac{2}{3}\right)^{-2}$
6. $\left(1\frac{3}{4}\right)^{\frac{1}{2}}$
7. $(-3.47)^{\frac{3}{5}}$
8. $(-2.49)^{\frac{-1}{2}}$
9. $\left(\frac{12.37}{2.69}\right)^{\frac{14}{17}}$
10. $(3.52^4 - 2.15^3)^{\frac{3}{2}}$
11. $(17.63)^{\frac{1}{2}} - (3.165)^{\frac{1}{3}}$
12. $(15.86)^{\frac{-3}{7}} + (-4.39)^{\frac{2}{7}}$
13. $\frac{(11.45)^{0.467}}{(2.78)^{-3.1}}$
14. $\frac{(-17.45)^{\frac{13}{27}}}{(15.35)^{\frac{3}{4}}}$
15. $(3.2)^{\frac{1}{3}} \times (2.5)^{\frac{1}{2}} - (1.36)^{\frac{1}{4}}$
16. $(-3.2)^{\frac{2}{3}}$
17. $(56)^{\frac{2}{3}}$
18. $(63)^{-\frac{3}{5}}$
19. $(-4.57)^{-\frac{3}{4}}$
20. $(-31.2)^{-\frac{4}{5}}$

Solution of Equations involving Indices

From an equation of the form $x^{\frac{m}{n}} = c$, x is found by raising the expression on each side of the '=' to the reciprocal of the power of x.

Example I: Solve $(x)^{\frac{15}{23}} = 26.4$ correct to 4 significant figures.

Solution: $(x)^{\frac{15}{23}} = 26.4$

$\therefore \left[(x)^{\frac{15}{23}}\right]^{\frac{23}{15}} = (26.4)^{\frac{23}{15}}$ [raising both sides to the **reciprocal power** to isolate the variable]

$\therefore x = 151.3$ [using a calculator]

Example J: Solve $(2A+3)^{\frac{-4}{7}} = 8$, to 3 significant figures.

Solution: $(2A+3)^{\frac{-4}{7}} = 8$

$\therefore \left[(2A+3)^{\frac{-4}{7}}\right]^{\frac{-7}{4}} = (8)^{\frac{-7}{4}}$ [raising to the reciprocal power]

$\therefore 2A + 3 = (8)^{\frac{-7}{4}}$

$\therefore 2A = (8)^{\frac{-7}{4}} - 3$ [subtracting 3]

$\therefore A = \dfrac{(8)^{\frac{-7}{4}} - 3}{2}$ [dividing by 2]

$\therefore A = -1.49$

Exercise 15e

1. Solve each of the following equations:

a. $x^{\frac{1}{2}} = 2$ b. $x^{\frac{1}{3}} = 2$ c. $x^{\frac{1}{4}} = 3$ d. $a^{\frac{1}{2}} = 5$

e. $a^{\frac{1}{5}} = 2$ f. $y^{-2} = 25$ g. $x^{-4} = 256$ h. $y^{\frac{3}{4}} = 27$

i. $x^{\frac{2}{5}} = 4$ j. $y^{\frac{4}{3}} = 16$ k. $x^{\frac{-5}{3}} = 32$ l. $y^{\frac{-7}{3}} = 128$

m. $2x^{\frac{1}{2}} - 1 = 3$ n. $4x^{\frac{1}{3}} + 1 = 9$ o. $2(x-1)^{\frac{1}{2}} = 4$ p. $\dfrac{x^{\frac{2}{3}} + 1}{2} = 5$

q. $\dfrac{3}{x^{\frac{1}{3}}} = 9$ r. $x^{\frac{-1}{2}} = 2$ s. $3y^{\frac{-3}{2}} - 1 = 80$ t. $\dfrac{z^{\frac{-2}{5}}}{3} - 1 = 2$

2. Solve the following equations:

a. $(x)^{\frac{2}{3}} = 16$ b. $(A)^{\frac{3}{7}} = 28$ c. $(A)^{\frac{-7}{3}} = 43$

d. $(p)^{1\frac{3}{4}} = 7$ e. $(p)^{\frac{4}{3}} = -2.3$ f. $(p)^{\frac{3}{5}} = -4$

g. $2(x)^{\frac{2}{3}} = 7$ h. $\left[(x)^{\frac{7}{9}} + 3\right]^{\frac{1}{2}} = 9$ i. $(2x-3)^{\frac{-3}{2}} = 5$

3. The relationship connecting length L to cost C of a type of material is given by $C = 5L^{\frac{2}{3}} + 6$, where C is given in dollars and L in metres.
   a. What is the cost when 27 metres are purchased?
   b. What length can be purchased for $100?

4. The relationship between the volume, height and base radius of a shape is given by $V = KR^{\frac{5}{2}}H^{\frac{1}{2}}$, where R is base radius, H is height and K is a constant. When the height and base radius are both 2 the volume is 16:
   a. Find the volume when the radius is 16 and the height 49.
   b. Find the height when the volume is 128 and the radius is 4.
   c. Find the radius when the volume is 486 and the height is 3 125.

Problems and Investigations

1. Investigate the claim that if A and B are whole numbers with A > B then $A^2 - B^2 > 1$ (hint: $A^2 - B^2 = (A-B)(A+B)$).

A	B	A – B	A + B	$A^2 - B^2$

2. Investigate the claim that if $x = (z+1)^{\frac{1}{2}} - (z-1)^{\frac{1}{2}}$ and $y = (z+1)^{\frac{1}{2}} + (z-1)^{\frac{1}{2}}$ then $xy = 2$ (hint: set up a table).

z	x	y	xy

Note: The above two investigations are particularly suitable for spreadsheet analysis.

16. LOGARITHMS

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 ALGEBRA

- *to carry out appropriate manipulation and simplification of algebraic expressions (including the concept, properties and manipulation of logarithms)*
- *to sketch graphs and investigate the graph of a function, using a calculator and plotting points if neccessary*
- *to use grapical methods to investigate a pattern in data and, where appropriate, identify its algebraic form*

The Natural Logarithm Function

The **natural logarithm** function has domain $\{x: x > 0\}$ and hence is defined for all positive numbers. The natural logarithm function has the symbol *l*n on calculators.

This function is also called the logarithm of x to the base e and is written $\log_e x$.

Example A: Calculate $\dfrac{(2 + \ln 34.3)^3}{3 - \ln 5.8}$

Solution: $\ln 34.3 = 3.5351454$ and $\ln 5.8 = 1.7578579$ [using a calculator]

$$\therefore \frac{(2 + \ln 34.3)^3}{(3 - \ln 5.8)} = \frac{(5.5351453)^3}{1.2421421}$$
$$= 136.5 \text{ (1 d.p.)}$$

Note: i. For accuracy, rounding off is left until the end of the calculation.
ii. $\ln^2 x$ means $(\ln x)^2$.

Exercise 16a

1. Calculate each of the following to 2 decimal places:

a. $\ln 25.6$	b. $\ln 136.8$	c. $\ln 42.5 + \ln 8.03$
d. $\ln 947.6 - \ln 84.5$	e. $2\ln 112.5$	f. $(\ln 36.7)(\ln 55.8)$
g. $(\ln 82.5)^2$	h. $\ln^2 (142.7)$	i. $\dfrac{\ln 5.3}{3}$
j. $\frac{1}{4}\ln 38.4$	k. $8 - \ln 2.6$	l. $17 \div \ln^3 (0 - 367)$
m. $(\ln 25.6 + 18)$	n. $\frac{1}{3}(\ln 5.7 + \ln 6.8)$	o. $\dfrac{3\ln 6.2 + 4}{4\ln 4.5 + 6}$
p. $\sqrt{6\ln 2.3 + 5\ln 4.3}$	q. $(\ln 6.4 + 3)^{\frac{2}{3}}$	r. $\dfrac{\ln 4.5 + \ln 2.7}{2^{\frac{1}{3}} - 1}$
s. $\dfrac{4}{(\ln 64)^{\frac{1}{3}}}$	t. $7^{0.4} - \ln (-2.3)$	
u. $(\ln 47.3)^{\frac{2}{3}} - (\ln 8.6)^{0.6}$	v. $\sqrt[5]{(\ln 8.7)^2 - 1}$	w. $\ln (\ln 4.8)$
x. $(5 - \ln 5)^{\frac{1}{4}}$	y. $\ln (2\ln 3 - \ln 9)$	

2. Calculate each of the following to 2 decimal places:

a. $\log_e 2.4$	b. $\log_e 5.7$	c. $\log_e 3.8$
d. $3\log_e 0.47$	e. $\log_e 0.56 + \log_e 2.1$	f. $\log_e 4.3 - \ln 4.3$
g. $2 + \ln 3.6 + \log_e 0.313$		
h. $(\log_e 0.32 + \ln 1.45)^2$	i. $\sqrt{\log_e 8.2}$	j. $\dfrac{1}{\log_e 7}$

Properties of the Natural Logarithmic Function

For all positive a and b:

$\ln a + \ln b = \ln ab$

Example B: $\ln 6 = 1.7917595$, $\ln 2 = 0.6931472$ and $\ln 3 = 1.0986123$.
It can be confirmed that $\ln 2 + \ln 3 = \ln (2 \times 3) = \ln 6$ as follows:
$$\ln 2 + \ln 3 = 0.6931472 + 1.0986123$$
$$= 1.7917595$$
$$= \ln 6$$

$\ln a - \ln b = \ln\frac{a}{b}$

Example C: $\ln 3 = 1.0986123$, $\ln 2 = 0.6931472$ and $\ln 1.5 = 0.4054651$.
It can be confirmed that $\ln 3 - \ln 2 = \ln \frac{3}{2} = \ln 1.5$ as follows:
$$\ln 3 - \ln 2 = 1.0986123 - 0.6931472$$
$$= 0.4054651$$
$$= \ln 1.5$$

$x\ln a = \ln a^x$

Example D: $\ln 2 = 0.6931472$ and $\ln 32 = 3.4657359$
It can be confirmed that $5\ln 2 = \ln 2^5$ as follows:

$$\begin{aligned} \ln 2 &= 0.6931472 \\ \therefore\ 5\ln 2 &= 5 \times 0.6931472 \quad \text{[multiplying by 5]} \\ &= 3.4657359 \\ &= \ln 32 \\ &= \ln 2^5 \end{aligned}$$

$\ln 1 = 0$ [The student can verify this for themselves.]

Exercise 16b

1. Express the following as a single natural logarithm [e.g. $\ln 3 + \ln 5 = \ln 15$]
   a. $\ln 4 + \ln 6$ b. $\ln 5 + \ln 6$ c. $\ln 2 + \ln 3 + \ln 4$
   d. $\ln 26 - \ln 2$ e. $\ln 36 - \ln 18$ f. $2\ln 5$
   g. $3\ln 4 - \ln 2$ h. $\frac{1}{2}\ln 36$ i. $\frac{1}{2}\ln 49 + \frac{1}{3}\ln 27$
   j. $\frac{1}{2}\ln 100 - \ln 2$

2. If $\ln a = x$, $\ln b = y$, $\ln c = z$, express the following in terms of x, y and z:
   [e.g. $\ln\left(\frac{ab^2}{c}\right) = \ln a + 2\ln b - \ln c = x + 2y - z$]
   a. $\ln\left(\frac{ab}{c^2}\right)$ b. $\ln\left(\frac{a^3b^2}{c}\right)$ c. $\ln\sqrt[3]{a}$ d. $\ln\left(\frac{a\sqrt[3]{b^2}}{c}\right)$

The Exponential Function

The **exponential function**, e^x, is closely related to the natural logarithmic function. It is its inverse. Its key is the inverse of the ln key on most scientific calculators.

Example E:

a. $e^{3.4} = 29.964$ [on a calculator: (3.4) → (inv) → (ln)]

b. $e^{\ln 7} = e^{1.9459101}$ [on a calculator: (7) → (ln) → (inv) → (ln)]
$= 7$

c. $\ln e^{4.3} = \ln 73.699793$
$= 4.3$

Note: Part b. and c. of example E show that the natural logarithm function and the exponential function 'cancel' each other out. This is because they are **inverse** functions of one another.

Properties of the Exponential Function

The exponential function follows all the rules associated with indices, such as:

i. $e^x.e^y = e^{x+y}$ ii. $e^x \div e^y = e^{x-y}$
iii. $(e^x)^y = e^{xy}$ iv. $\frac{1}{e^x} = e^{-x}$
v. $e^0 = 1$

Example F:

i. $e^3 = 20.085537$ $e^2 = 7.3890561$ $e^5 = 148.41316$
$e^3.e^2 = 148.41316 = e^5$

ii. $\frac{1}{e^2} = \frac{1}{7.3890561} = 0.1353352 = e^{-2}$

Exercise 16c

1. Calculate each of the following to 3 decimal places:
   a. $e^{1.3}$ b. $e^{2.8}$ c. $e^{-3.1}$ d. $e^{-1.13}$
   e. $2e^{1.4}$ f. $e^{4.3} + e^{0.32}$ g. $e^{5.6} - e^{4.8}$ h. $e^{3.4} - 8$
   i. $7 - e^{0.32}$ j. $\frac{1}{e^3}$ k. $\frac{2}{e^2}$ l. $e^2 - \frac{1}{e^1}$
   m. $e^{\frac{1}{3}}$ n. $e^{\frac{3}{4}}$ o. $e^{-\frac{1}{2}}$

2. Calculate each of the following to 2 decimal places:
   a. $e^{0.9}e^{0.4}$ b. $e^{-2.4}$ c. $\sqrt[3]{e}$ d. $e^{\sqrt{5}}$
   e. $(e^{1.3} - 1)^2$ f. $e^{3.4} - e^{1.2}$ g. $(e^{4.1} + 3)^{\frac{2}{5}}$ h. $\frac{3e^{1.3} - 4}{2e^{1.2} - 1}$
   i. $\frac{(e^2 - 0.1)^2}{1.3^{\frac{3}{2}}}$ j. $\frac{2.3^{\frac{4}{5}} - e^{-3.1}}{e^{\sqrt{2}}}$

3. The function $\exp(x)$ is often used to represent e^x. Sometimes it is written $\exp x$. Thus $\exp(2) = \exp 2 = e^2$.
   Calculate the following to 4 decimal places:
   a. $\exp 3.7$ b. $\exp 4.03$ c. $\exp(2.73)$ d. $\exp(-6.34)$
   e. $\exp(-0.34)$

4. Calculate each of the following to a maximum of 2 decimal places:
   a. $e^{\ln 5}$ b. $e^{\ln(2^{\frac{1}{2}} - 1)}$ c. $\ln e^{\sqrt{3}}$ d. $3\ln e^{\frac{1}{\sqrt{7}}} - 1$ e. $e^{\ln 5} - 3e^0$

5. Simplify each of the following expressions as much as possible.
   a. $e^a.e^b$ b. $e^x.e^2$ c. $e^x.e^{-3}$ d. $e^{2x} \div e^x$ e. $e^{3x} \div e^3$
   f. $\frac{e^y}{e^x}$ g. $(e^x)^2$ h. $\left(\frac{1}{e^x}\right)^3$ i. $e^x.e^{-2}$ j. $(e^x.e^y)^2 \div e^x$

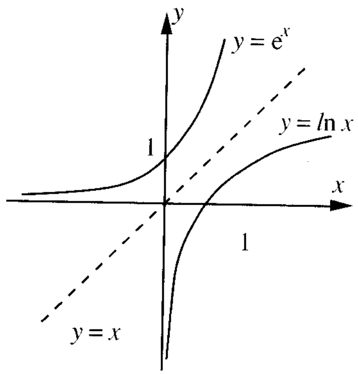
The Graphs of the Logarithmic and Exponential Function

The tables below give selected points on the graphs of $y = ln\,x$ and $y = e^x$.

x	0.1	0.5	1	2	5	10
$ln\,x$	-2.3	-0.7	0	0.7	1.6	2.3

x	-2	-1	0	1	2	3
e^x	0.1	0.4	1	2.7	7.4	20.1

Plotting them on the same diagram and sketching the graphs yields the following diagram:



Clearly the graphs of $y = ln\,x$ and $y = e^x$ are mirror images of each other in the line $y = x$, which is the result of the two functions being inverses of each other.

Exercise 16d

Sketch the following graphs:

1. $y = 2ln\,x$
2. $y = ln(x - 2)$
3. $y = 1 + ln\,x$
4. $y = 3e^x$
5. $y = e^{x+1}$
6. $y = e^{-x}$
7. $y = ln\,(-x)$

Solving Logarithmic and Exponential Equations

Because the logarithmic and exponential functions are inverses of each other the following important relationship holds:

$y = ln\,x$ if and only if $e^y = x$

This provides the key to solving many equations involving these functions.

Example G: Solve: i. $ln\,x = 3$ ii. $e^x = 4$.

Solution: i. $ln\,x = 3$

$\therefore x = e^3$ [as $ln\,x = y$ if, and only if, $x = e^y$]

$\therefore x = 20.1$ (1 d.p.)

[**NB:** The step in going from $ln\,x = 3$ to $x = e^3$ is called 'switching to exponentials'.]

ii. $e^x = 4$

$\therefore x = ln\,4$ [as $e^y = x$ if, and only if, $y = ln\,x$]

[**NB:** The step in going from $e^x = 4$ to $x = ln\,4$ is called 'switching to logarithms'.]

Example H: Solve: i. $e^{3x+1} = 7$ ii. $ln\,\frac{x+3}{2} = 0.1$

Solution: i. $e^{3x+1} = 7$

$\therefore 3x + 1 = ln\,7$ [switching to logarithms]

$\therefore x = \frac{ln\,7 - 1}{3}$ [rearranging]

$= 0.315$ (3 d.p.) [solving the equation]

ii. $ln\,\frac{x+3}{2} = 0.6$

$\therefore \frac{x+3}{2} = e^{0.6}$ [switching to exponentials]

$\therefore x = 2e^{0.6} - 3$

$= 0.644$ (3 d.p.) [solving the equation]

Exercise 16e

1. Solve the following equations to 2 decimal places:
   a. $e^x = 4$ b. $e^x = 0.3$ c. $e^{x-1} = 1.2$
   d. $e^{3x+4} = 5$ e. $e^{\frac{x-1}{3}} = 0.6$ f. $e^{-x} = 3.4$
2. Solve the following equations to 2 decimal places:
   a. $ln\,x = 3$ b. $ln\,x = -2$ c. $ln\,(x + 1) = -2$
   d. $ln\left(\frac{8x-3}{4}\right) = 4$ e. $ln\left(\frac{x+1}{3x-1}\right) = \frac{1}{2}$

Other Logarithmic Functions

Any function of the type $y = a^x$ is called an **exponential function**. The inverse of this function is called the 'logarithmic function to base a' and is written $\log_a x$. Most scientific calculators have the logarithmic function to base 10 and the exponential function 10^x in addition to the natural logarithmic function $\ln x$ and the exponential function e^x. On most calculators the logarithmic function to base 10 has the symbol 'log' and the exponential function 10^x has the symbol '10^x'.

Example I:

a. $\sqrt{\log 17} = \sqrt{1.2304489}$
$= 1.11$ (2 d.p.)

b. $(10^{1.2} - 1)^{\frac{1}{3}} = (15.848932 - 1)^{\frac{1}{3}}$
$= 2.46$ (2 d.p.)

c. $10^{\log 5} = 5$ [as '10^x' and 'log' are inverse functions]

d. $10^{2x} = 6$
$\therefore\ 2x = \log 6$ [taking logs]
$\therefore\ x = 0.39$ (2 d.p.)

Exercise 16f

Calculate to 2 decimal places using the 10^x and log keys on your calculator:

1. $\log 125.7$
2. $10^{2.3}$
3. $10^{3.4} - \log 2347.5$
4. $\dfrac{10^{-2.3}}{\log 1.37}$
5. $10^{\log 3.678}$

General Properties of Logarithmic Functions

a. If a is any positive number, $y = a^x$ and $y = \log_a x$ are **inverse functions**.
Note: $\log_a x$ is the logarithm of x to the base *a*.

b. The graph of $y = a^x$ is similar to that of $y = e^x$ and the graph of $y = \log_a x$ is similar to that of $y = \ln x$. The graphs of $y = a^x$ and $y = \log_a x$ are **images** of each other under reflection in the line $y = x$.

c. The function $y = \log_a x$ is defined by:

$y = \log_a x$ if and only if $x = a^y$

d. $a^{\log_a x} = x$

$a^{\log_a x} = x$ because a^x and $\log_a x$ are inverse functions of each other.

e. $\log_a a^x = x$

f. For all $a > 0$: $\log_a a = 1$ and $\log_a 1 = 0$

g. $\log_a(xy) = \log_a x + \log_a y$

Proof: Let $u = \log_a x$ and $v = \log_a y$
$\therefore\ x = a^u$ and $y = a^v$ [from definition of log]
$\therefore\ xy = a^u a^v$ [multiplying x and y]
$\therefore\ xy = a^{u+v}$ [adding indices]
$\therefore\ u + v = \log_a xy$ [by definition of log]
$\therefore\ \log_a x + \log_a y = \log_a xy$ [since $u = \log_a x$, $v = \log_a y$]
Q.E.D.

h. $\log_a\left(\frac{s}{t}\right) = \log_a s - \log_a t$

i. $\log_a x^n = n \log_a x$

Worked Examples

Example J: Write $128 = 2^7$ in logarithmic form.

Solution: $128 = 2^7$
$\therefore\ 7 = \log_2 128$ [since $y = \log_a x$ when $x = a^y$]
$\therefore$ The required expression is $\log_2 128 = 7$.

Example K: Solve the equation $5^{3t-1} = 8$.

Solution: $5^{3t-1} = 8$
$\therefore\ \ln 5^{3t-1} = \ln 8$ [taking logs of both sides]
$\therefore\ (3t - 1)\ln 5 = \ln 8$ [since $\log A^n = n\log A$]
$\therefore\ 3t - 1 = \dfrac{\ln 8}{\ln 5}$ [dividing by $\ln 5$]
$\therefore\ 3t = \dfrac{\ln 8}{\ln 5} + 1$ [adding 1]
$\therefore\ t = \left(\dfrac{\ln 8}{\ln 5} + 1\right) \div 3$
$= 0.764$ (3 d.p.)

Note: a. $\ln 8$ and $\ln 5$ are evaluated last to minimise numerical errors.
b. Using the 'log' key instead of the '$\ln$' key gives the same result.

Example L: To find the value of $\log_{16}15$,

let $x = \log_{16}15$.

$\therefore\ 16^x = 15$ [since $x = a^y$ when $y = \log_a x$]

$\therefore\ \ln 16^x = \ln 15$ [taking logs of both sides]

$\therefore\ x \ln 16 = \ln 15$ [since $\log A^n = n \log A$]

$\therefore\ x = \dfrac{\ln 15}{\ln 16}$

$\therefore\ x = 0.977$ (3 d.p.)

Note: The **base changing rule** is: $\log_a x = \dfrac{\log_b x}{\log_b a}$

Example M: Solve $\log_3(2x + 1) = 5$.

Solution: $\log_3(2x + 1) = 5$

$\therefore\ 2x + 1 = 3^5$ [$x = a^y$ when $y = \log_a x$]

$\therefore\ 2x + 1 = 243$ [since $3^5 = 243$]

$\therefore\ 2x = 242$ [subtracting 1]

$\therefore\ x = 121$ [dividing by 2]

Example N: Express $2\log_a x - 3\log_a y + 4$ as a single logarithm.

Solution: $2\log_a x - 3\log_a y + 4$

$= \log_a x^2 - \log_a y^3 + 4$ [$n \log u = \log u^n$]

$= \log_a \dfrac{x^2}{y^3} + 4$ [$\log u - \log v = \log\dfrac{u}{v}$]

$= \log_a \dfrac{x^2}{y^3} + 4\log_a a$ [$\log_a a = 1$]

$= \log_a \dfrac{x^2}{y^3} + \log_a a^4$ [$n \log u = \log u^n$]

$= \log_a \dfrac{x^2 a^4}{y^3}$ [$\log u + \log v = \log uv$]

Exercise 16g

1. Sketch the following graphs:
   a. $y = 2^x$ b. $y = \log_2 x$ c. $y = 2^{x+1}$ d. $y = 2^{1-x}$
   e. $y = 2 - 2^x$ f. $y = \log_2 (x + 1)$ g. $y = 3^x$ h. $y = \log_3 x$

2. Write the equivalent logarithmic statement for each of the following:
   a. $2^3 = 8$ b. $3^4 = 81$ c. $5^2 = 25$ d. $6^3 = 216$ e. $x^a = b$

3. Write the equivalent exponential statement for the following:
   a. $\log_3 9 = 2$ b. $\log_9 3 = \dfrac{1}{2}$ c. $\log_{25}125 = 1.5$
   d. $\log_{1000}10 = \dfrac{1}{3}$ e. $\log_3\dfrac{1}{3} = -1$

4. Solve the following for x:
   a. $2^x = 64$ b. $2^x = \dfrac{1}{2}$ c. $2^{x-1} = 32$ d. $2^x = 128$
   e. $3^x = \dfrac{1}{3}$ f. $3^x = \dfrac{1}{27}$ g. $(3)^{\frac{1}{x}} = 9$ h. $125 = 5^x$
   i. $9^x = 3$ j. $16^x = 2$ k. $25^x = 5$ l. $25^x = 125$
   m. $25^x = 625$ n. $64^x = 32$ o. $49^x = \dfrac{1}{7}$ p. $121^x = 11^2$
   q. $2^{1-x} = 4$ r. $\dfrac{1}{2^x} = 8$ s. $27^x = 243$ t. $81^x = 27$

5. Solve the following equations to 3 decimal places:
   a. $10^x = 9$ b. $10^{-x} = 131$ c. $e^x = 12.3$
   d. $e^x = 17.9$ e. $e^{3x} = 18.45$ f. $3^x = 4$
   g. $2^{3x-1} = 9$ h. $3^x = 2^{x+1}$ i. $4^{2x} = 5^{x+2}$
   j. $3.6^{4x+5} = 2.9^{2x+13}$

6. Calculate the following:
   a. $\log_4 4$ b. $\log_3 3$ c. $\log_2 8$ d. $\log_9 3$
   e. $\log_{16}2$ f. $\log_{16}32$ g. $\log_{625}125$ h. $\log_{196}14$
   i. $\log_{32}16$ j. $\log_{0.1}10$

7. Solve the following equations for x:
   a. $\log_2 x = 6$ b. $\log_3 x = 4$ c. $\log_4 x = \dfrac{1}{2}$
   d. $\log_6 x = 3$ e. $\log_x 3 = 2$ f. $\log_x 4 = \dfrac{1}{3}$
   g. $\log_x 9 = \dfrac{1}{2}$ h. $\log_x 3 = 1$ i. $\log_x 16 = 4$
   j. $\log_x 216 = \dfrac{3}{2}$ k. $\log_2 32 = x$ l. $\log_3 27 = x + 2$
   m. $\log_{\frac{1}{4}} 64 = x$ n. $\log_5 25 = 3x + 1$ o. $\log_8 4 = x$

8. Express each of the following as a single logarithm:
   a. $\log x + \log y + \log z$ b. $2 \log x + \log y$
   c. $\log x + \log y + \log y$ d. $\log x + \log y - \log z$
   e. $\log a + 3 \log b - 2 \log c$ f. $\dfrac{1}{2} \log p - \dfrac{3}{2} \log q$
   g. $\log xy + \log z$ h. $\log xyz - \log x - \log z$
   i. $\log x^2y^3 - \log x + \log y$ j. $\log\left(\dfrac{xy}{z}\right) + \log\left(\dfrac{z}{x}\right)$
   k. $2 \log\left(\dfrac{x}{y}\right) - \log x^2$ l. $\log\left(\dfrac{1}{x}\right) + \log x + 2 \log y$
   m. $\dfrac{1}{2} \log x - \log \sqrt{x} + \log xy$ n. $\log \dfrac{x}{yz} + \log \dfrac{1}{z} - \log \dfrac{x}{y}$
   o. $2 \log xy - \log \dfrac{x}{y}$ p. $2 \log_a x + 5$

9. Express the following as logarithms of single numbers:
   a. $\log 12 + \log 5$ b. $\log 60 - \log 5$
   c. $2 \log 5 + \log 3$ d. $\frac{1}{2}\log 9 + \frac{2}{3}\log 8$
   e. $\frac{3}{2}\log 4 + \frac{1}{4}\log 81$ f. $\frac{1}{3}\log 27 + \frac{1}{5}\log 32 - \log 7$
   g. $\frac{2}{5}\log 32 - \log 4$ h. $3 \log 5 - \log 15$
   i. $7 \log 2 - \frac{1}{2}\log 16 + \log 3$ j. $\log 15 + \log 20 - \log 25$

10. Simplify to a single number given as a fraction or integer without using a calculator:
   e.g. $\frac{\log 7}{\log 49} = \frac{\log 7}{2\log 7} = \frac{1}{2}$
   a. $\frac{\log 8}{\log 2}$ b. $\frac{\log 25}{\log 125}$ c. $\frac{\log 36}{\log 216}$ d. $\frac{\log 625}{\log 125}$
   e. $\frac{\log 64}{\log 32}$ f. $\frac{\log 196}{\log 14}$ g. $\frac{\log 64 + \log 32}{\log 32}$
   h. $\frac{\log 7 + \log 7}{\log 49}$ i. $\frac{\log 32 - \log 8}{\log 4}$ j. $\frac{\log 625 + \log 25}{\log 125}$

Problems and Investigations

1. The population p of a species of creatures after t years is given by $\frac{6e^t - 2}{2e^t + 5}$ where p is measured in millions.
   a. What is the initial population?
   b. What is the population after 3 years?
   c. After how many years is the population equal to 2 million?
   d. Sketch the graph of population against time.
   e. What size does the population get close to as the years pass?

2. The number of deer in a forest after t years is $N = 2000 - 1000e^{-t}$.
   a. How many deer are there in the forest when records are first collected?
   b. How many deer are there in the forest $3\frac{1}{2}$ years later?
   c. Sketch the graph of population against time.
   d. What is the maximum number of deer which can live in the forest?
   e. At what time is the population of deer equal to 1 500?

3. The cost of a hectare of land is believed to be Ae^{kt} in dollars, where t is the number of years, and A and k are constants. After 3 years the value of the land is \$112 486 and after 7 years it is \$131 593.
   a. Find the values of A and k.
   b. Find the value of the land after 10 years.
   c. When will the land have a value of \$1 000 000?

4. The relationship between length (L) and area (A) of a rectangular piece of material is given by $\frac{2}{3}\ln A = \ln K + \frac{1}{3}\ln L$. K is a constant.
   a. Find K if A=27 when L = 8.
   b. Make A the subject of the formula.
   c. Sketch the graph of A against L.
   d. Find the value of L when the area is 64.
   e. Find an expression for the width of the rectangle in terms of L.

5. The number of particles of ash deposited up a chimney is given by $\ln N = A\ln h + \ln k$, where N is the number of particles, h is the height of the chimney, and A and k are constants. At a height of 1 metre the number of particles deposited is 1 000. At a height of 5 metres the number of particles deposited is 25 000.
   Find the height of the chimney if 100 000 particles are deposited.

17. SEQUENCES AND SERIES

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 ALGEBRA

- *to use sequences and series to model real problems and interpret their solutions*
- *to describe and use arithmetic and geometric sequences or series in common situations*

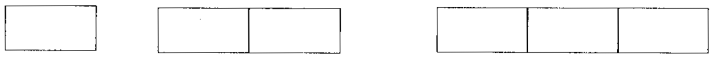
Introduction

A **sequence** is an ordered list of numbers. Unless otherwise stated it is always taken to be infinite. This means it goes on forever. A sequence associates a real number with each natural number.

The real number associated with the natural number 1 is called the *first* **term**.
The number associated with 2 is called the *second* term and so on.
The number associated with n is called the **nth term** or **general term**.

Example A: The infinite set of numbers: 1, 3, 5, 7, ... is a sequence with a **1st** term of 1, 2nd term of 3, and 3rd term of 5. The general or nth term is 2n – 1.

Example B: Consider the following pattern of rectangles:



I II III

Write down the first four terms and the general term of the sequences of:
a. the number of rectangles.
b. the number of vertices (points).
c. the number of sides.

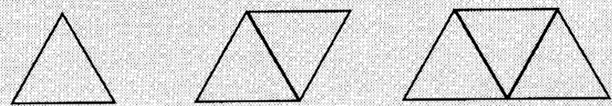
Solution:
a. 1, 2, 3, 4, . . . , n, . . .
b. 4, 6, 8, 10, . . . , 2n + 2, . . .
c. 4, 7, 10, 13, . . . , 3n + 1, . . .

Exercise 17a

1. For each of the following sequences, write down the next 2 terms and the general term.

a. 2, 3, 4, 5, . . .
b. 0, 1, 2, 3, . . .
c. 3, 6, 9, 12, . . .
d. 3, 5, 7, 9, 11, . . .
e. -3, -1, 1, 3, . . .
f. 1, 2, 4, 8, . . .
g. 1, 3, 9, 27, . . .
h. 2, 6, 18, 54, . . .
i. $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$
j. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
k. $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$
l. 10, 8, 6, . . .
m. 16, 8, 4, 2, . . .
n. 1.00, 0.50, 0.33, 0.25, . . .
o. 1, 4, 9, 16, . . .
p. 3, 6, 9, 12, . . .
q. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$
r. 1, 3, 5, 7, 9, . . .
s. 2, 5, 10, 17, . . .
t. 1.00, 1.41, 1.73, 2.00, . . .

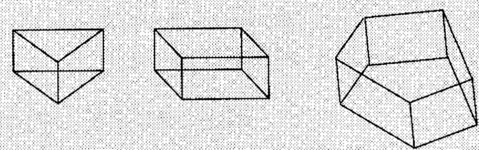
2. Consider the following pattern of triangles:



I II III

Write down the first 4 terms and the general term of the sequences of:
a. the number of triangles.
b. the number of vertices.
c. the number of sides.

3. Consider this pattern of shapes:



I II III

Write down the first 4 terms and the general term of the sequences of:
a. the number of vertices or points.
b. the number of faces.
c. the number of edges.

Listing Sequences

Sequences are not usually represented in the mapping notation used for real numbers. Thus the sequence given in mapping notation by $f: n \to 2n - 1, n \in N$ is usually written in one of the following ways:

a. **Enumeration or listing:**
$1, 3, 5, 7, ..., 2n - 1, ...$

b. Writing down the nth term:
in **triangular brackets** $< 2n - 1 >$ or by **formula**, $T_n = 2n - 1$.

c. **Recursively**:
The first term(s) is given and the remaining terms are found from preceding terms by a formula. e.g., $U_1 = 1$ and $U_{n+1} = U_n + 2$.

Example C: List the first four terms of each of the following sequences:
a. $< n^3 - n^2 >$ b. $T_n = 3n^2 - n$ c. $A_1 = 2, \quad A_{n+1} = A_n + n$

Solution: In each case the terms are worked out by substitution.

a. The first term is $1^3 - 1^2 = 0$ [substituting $n = 1$ in $n^3 - n^2$]
The second term is $2^3 - 2^2 = 4$ [substituting $n = 2$ in $n^3 - n^2$]
The third term is $3^3 - 3^2 = 18$ [substituting $n = 3$]
The fourth term is $4^3 - 4^2 = 48$ [substituting $n = 4$]
$\therefore$ The first four terms are 0, 4, 18 and 48.

b. The first four terms are: $3 \times 1^2 - 1$, $3 \times 2^2 - 2$, $3 \times 3^2 - 3$ and $3 \times 4^2 - 4$
$\therefore$ The terms are 2, 10, 24 and 44.

c. $A_1 = 2$ [given]
$A_2 = A_{1+1} = A_1 + 1 = 3$ [substituting $n = 1$ into $A_{n+1} = A_n + n$]
$A_3 = A_{2+1} = A_2 + 2 = 5$ [substituting $n = 2$]
$A_4 = A_{3+1} = A_3 + 3 = 8$ [substituting $n = 3$]
$\therefore$ Terms are 2, 3, 5 and 8

Exercise 17b

1. For each of the sequences given by the following formulas, write down the first 4 terms:
   a. $<n^2 + 1>$ b. $<2n + 3>$ c. $A_n = n^3$
   d. $A_n = n^2 + n + 1$ e. $<\frac{n^2 + 1}{n}>$ f. $A_n = (1 + \frac{1}{n})^n$
   g. $A_n = \frac{(-1)^n n}{2^n}$ h. $<n^2 + (-1)^n>$

2. Write down the first four terms of these sequences:
   a. First term is 30, each term thereafter is 40% larger than its predecessor.
   b. First term is 12, each term is 10% larger than its predecessor.
   c. Second term is 48, each term is bigger by 7 than its predecessor.
   d. Each term is the square of the one before it. The fourth term is 16.
   e. Each term is double the one before it. First term is 6.

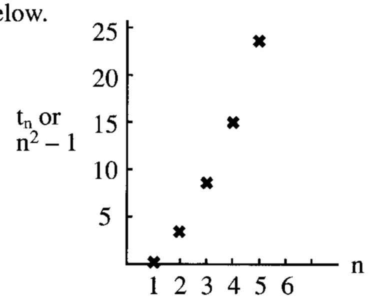
3. The following sequences are defined recursively. What are the first 4 terms?
   a. $A_1 = 1, A_n = 2A_{n-1}$ b. $A_1 = 1, A_n = A_{n-1} + 4$ c. $A_1 = 3, A_n = (A_{n-1})^2$
   d. $A_1 = 1, A_n = \frac{1}{n}A_{n-1}$ e. $A_1 = 1, A_n = 2A_{n-1} + 1$

The Graph of a Sequence

Example D: Draw the graph of the sentence $<n^2 - 1>$.

Solution: Listing the sequence we have 0, 3, 8, 15, 24, . . .

The graph appears below.



Notice that the graph appears as a pattern of points corresponding to the natural numbers. The graphs of all sequences have this feature.

Use of Graphing Calculators

Most calculators with graph-drawing capabilities make the drawing of the graphs of sequences very simple . For Casio calculators such as the *fx*-7700 GB and related models, here is a simple program which will plot the graph of the sequence n^2:

$$? \to n : \text{plot } n, n^2$$

Repeated pressing of the (EXE) key and entry of 1, 2, 3, . . . in that order will result in the rapid drawing of the graph. Users of other brands should consult their owner's manuals.

Exercise 17c

Draw the graphs of the following sequences:

1. $<2n + 1>$ 2. $<3n - 1>$ 3. $T_n = 2n + 2$ 4. $<n^2 + 1>$
5. $<2^n>$ 6. $T_n = 1 + \frac{1}{n}$ 7. $<2 - \frac{1}{n}>$
8. $A_1 = 2, A_n = A_{n-1} + 3$

Limit of a Sequence

A sequence $<U_n>$ has a **limit** if there is a number L which the terms of the sequence get closer and closer to as n tends to infinity. If $<U_n>$ has a limit and the limit is written L, then:

$\lim_{n \to \infty} U_n = L$ or $U_n \to L$ as $n \to \infty$	which is read 'U_n tends to L as *n* tends to infinity.'

Example E: Show that the sequence $\frac{3n+1}{2n-2}$ has a limit as n tends to infinity and find the limit.

Solution: Using a calculator and substituting gives:
$T_{100} = 1.5202$, $T_{1000} = 1.502002$ and $T_{10000} = 1.5002$
As n increases the terms get closer and closer to 1.5. The limit is thus 1.5 and can be written $T_n \to 1.5$ as $n \to \infty$.

Example F: A person on a diet loses 5kg the first week, 2.5kg the next week, 1.25kg the next week, etc. His weight was initially 105kg.

a. Write down his body weight at the end of the first 4 weeks.

b. His bodyweight after n weeks on the diet is $95 + \frac{10}{2^n}$. What weight does he get closer and closer to?

Solution: a.
$105 - 5 = 100$
$100 - 2.5 = 97.5$
$97.5 - 1.25 = 96.25$
$96.25 - 0.625 = 95.625$
Bodyweight at the end of the fourth week is 95.625kg

b. setting n = 50 and n = 100 we find that his bodyweight gets closer and closer to 95 which is the limit of the sequence $95 + \frac{10}{2^n}$.

Exercise 17d

1. Find the limits as $n \to \infty$, if they exist, of the following sequences:
   a. $<2n>$ b. $<\frac{2}{n}>$ c. $<1 + \frac{1}{n}>$ d. $<\frac{2^n}{3^n}>$
   e. $<\frac{n}{2^n}>$ f. $A_n = (n)^{\frac{1}{2}}$ g. $A_n = (n)^{\frac{1}{n}}$
   h. $A_n = \frac{3^n + 1}{2n + 1}$ i. $T_n = \frac{2 - n}{1 - 2n}$ j. $L_n = \frac{n^2 + n + 1}{2n^2 - n + 2}$
2. A person on a diet loses 9kg the first week, 6kg the next, 4kg the next. Each week she loses $\frac{2}{3}$ of the amount she lost in the previous week. Initially she was 100kg.
   a. Write down her weight at the end of each of the first 6 weeks.
   b. What is the limit of the sequence of weekly weights?
   c. What does this limit represent in practical terms?
3. A company employs 2 workers who it pays a total of $50 an hour. Each new worker added to the payroll thereafter, gets paid $8 an hour.
   a. What is the total number of workers if n new workers have been added?
   b. What is the total hourly wage bill if n new workers have been added?
   c. What is the average hourly wage if n new workers have been added?
   d. Explain why the sequence of average hourly wage rates has a limit of $8 an hour, in practical terms.

Series

A **series** is obtained by *adding the successive terms of another sequence.*

Example G: 1 + 2 + 3 + 4 + ... is an infinite series
1 + 2 + 3 + 4 + 5 is a finite series

Sigma Notation

The Greek letter Σ (sigma) is used to indicate a summation.

a. $\sum_{i=1}^{5} a_i$ means $a_1 + a_2 + a_3 + a_4 + a_5$ b. $\sum_{n=3}^{7} u_n$ means $u_3 + u_4 + u_5 + u_6 + u_7$

Example H: $\sum_{n=1}^{4} 2n + 1 = (2 \times 1 + 1) + (2 \times 2 + 1) + (2 \times 3 + 1) + (2 \times 4 + 1)$
$= 3 + 5 + 7 + 9$
$= 24$

Example I: Express 2 + 4 + 6 + 8 + 10 in sigma notation.
$2 + 4 + 6 + 8 + 10 = 2 \times 1 + 2 \times 2 + 2 \times 3 + 2 \times 4 + 2 \times 5$
$= \sum_{j=1}^{5} 2j$

Exercise 17e

1. a. $\langle a_n \rangle$ is the sequence 1, 3, 1, 3, ... Find $\sum_{n=1}^{7} a_n$

   b. $\langle b_n \rangle$ is the sequence 1, 1.1, 1.11, 1.111, ... Find $\sum_{n=1}^{5} b_n$

   c. Find $\sum_{n=3}^{5} \left(\frac{1}{n}\right)$ d. Evaluate $\sum_{n=1}^{16} (-1)^n$

   e. $T_n = n^2 + 1$. Evaluate $\sum_{j=1}^{4} T_j$ f. $L_n = (n+1)^2 - n^2$. Calculate $\sum_{n=2}^{6} L_n$

   g. Evaluate $\sum_{n=1}^{7} n^2 - \sum_{n=1}^{6} n^2$ h. Evaluate $\sum_{n=1}^{50} n^2 - \sum_{n=1}^{49} n^2$

2. Write the following using sigma notation:

   a. $3 + 6 + 9$ b. $4 + 6 + 8$ c. $5 + 6 + 7 + 8$

   d. $1 + 4 + 9 + 16$ e. $4 + 8 + 16 + 32$

Arithmetic Sequences

An **arithmetic sequence**, often called an **arithmetic progression** (A.P.), is a sequence in which each term after the first differs from the previous term by a constant difference. This difference is called the **common difference** or just the *difference*.

Example J: 2, 5, 8, 11 and 14 are the first five terms of an arithmetic sequence because each subsequent term, 5, 8, 11 and 14 differs by 3 from the term before. $5 - 2 = 3$, $8 - 5 = 3$, $11 - 8 = 3$ and $14 - 11 = 3$; thus the sequence is arithmetic.

If the first term of an arithmetic progression is A_1 and the difference is d, then the the sequence, when enumerated, is: $A_1, A_1 + d, A_1 + 2d, A_1 + 3d, \ldots,$

Since the number of the d's is always *one less* than the number of the term, the nth term is always:

$$A_n = A_1 + (n-1)d$$

Example K: Janette is saving money and aims to increase the amount of money she deposits in her savings account by $10 each month. The first month she starts with a deposit of $25.00.

a. Write down the amounts she deposits during the first five months.

b. How much will Janette be depositing exactly 2 years after she started?

Solution:

a. Amounts deposited are $25, $35, $45, $55, $65 . . .

b. The sequence is an A.P. with $A_1 = 25$ and $d = 10$

$\therefore A_n = 25 + (n-1)\,10$ [substituting $A_1 = 25$ and $d = 10$]

$\therefore$ Amount deposited $= 25 + (24 - 1)10$ [after 2 years, n = 24]

$= \$255$

Example L: The fifteenth term of an arithmetic progression is 50 and the sixtieth term is 185. Find the first term and the common difference.

Solution: Let the first term be A_1 and the difference d. Using the formula and substituting in $A_n = A_1 + (n-1)\,d$ gives:

$A_1 + 14d = 50$ ....(1)

$A_1 + 59d = 185$ ....(2)

$\therefore 45d = 135$ [solving simultaneously, (2) – (1)]

$\therefore d = 3$

$A_1 + 42 = 50$ [substituting for d in (1)]

$\therefore A_1 = 8$

Exercise 17f: Arithmetic Sequences and Series

The following sequences are A.P.s; find the indicated terms:

1. a. 2, 5, 8, 11, ... T_6, T_{20}, T_n b. 11, 15, 19, 23, ... T_7, T_{11}, T_n

   c. 4, 2, 0, –2, ... A_6, A_{20}, A_n d. 1, 1.5 , 2, ...A_6, A_{12}, A_n

   e. 2, 2.1, 2.2, 2.3, ... A_6, A_{12}, A_n

2. a. In an A.P. $T_3 = 4$, $T_4 = 8$. Write down the first 6 terms and the nth term.

   b. In an A.P. $T_5 = 6$, $T_6 = 11$. Write down the first 8 terms and the nth term.

   c. In an A.P. $T_4 = 5$, $T_7 = 11$. Write down the first 8 terms and the nth term.

   d. In an A.P. $T_3 = 6$, $T_6 = -3$. Write down the first 8 terms and the nth term.

   e. 11 is the 5th term of an A.P. and -17 is the 12th term. Find the 58th term.

   f. $A_7 = 210$ in an A.P. with a difference of 7. Which term is 448?

   g. In an A.P. with difference of 8, the 4th term is 35. What term is 115?

   h. The 12th term of an A.P. is 29. If the 3rd term is subtracted from the 7th term we obtain -8. Find the 52nd term.

   i. The 15th term of an A.P. is 219 and the difference is 3. How many terms are there before the terms of the sequence exceed 1 000?

   j. What is the common difference of an A.P. whose difference is half the first term and whose 12th term is 169?

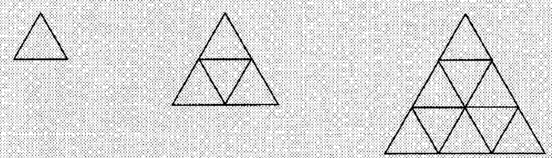
3. Stephen starts with 100 phonecards in his collection in January 1992 and increases the number by 3 each month. In January 1992 Stephanie starts with 200 and increases the number by 2 each month.

   a. How much time will elapse before Stephen has more than 500 cards?

   b. i. Will Stephanie ever have 240 cards?

      ii. If so, at the end of what month?

   c. When will Stephen first have more than Stephanie?

4. 

   This diagram shows a sequence of pictures made by successively adding layers of triangles.

L_n is the number of triangles in the bottom layer of the nth picture.
J_n is the number of triangles in the nth picture.
S_n is the number of sides in the bottom layer of the nth picture.
P_n is the number of points or vertices in the bottom layer of the nth picture.
W_n is the total number of points or vertices in the nth picture.

a. Draw the next diagram in the sequence.
b. Enumerate the first four terms of each of the sequences $\langle L_n \rangle$, $\langle J_n \rangle$, $\langle S_n \rangle$, $\langle P_n \rangle$, $\langle W_n \rangle$. Find the general term for each.
c. Which are arithmetic?

The Sum of the First *n* Terms of an A.P.

The sum of the first n terms of an arithmetic progression in which the first term is A_1 and the difference d is:

$$S_n = \frac{n}{2}[2A_1 + (n-1)d] \text{ or } S_n = \frac{n}{2}(A_1 + A_n)$$

Example M: A company increases the value of goods it exports each year by \$86 000. In 1977 it exported \$975 000 worth of goods. What will the total value be of all goods exported during the ten years beginning in 1977?

Solution: Annual exports are an A.P. with $A_1 = 975\,000$ and $d = 86\,000$.

Using the formula $S_n = \frac{n}{2}[2A_1 + (n-1)d]$ gives, for 10 years:

$$S_{10} = 5[2 \times 975\,000 + (10-1)\,86\,000]$$
$$= \$13\,620\,000$$

Example N: How many terms of the A.P. with first term 1 and difference of 4 should be added together to give a sum of 153?

Solution: Let the number of terms be N.

$$S_N = \frac{N}{2}[2A_1 + (N-1)d]$$
$$153 = \frac{N}{2}[2 + (N-1)4] \quad \text{[substituting } A_1 = 1, d = 4, S_N = 153]$$
$$306 = N[2 + (N-1)4] \quad \text{[multiplying by 2]}$$
$$= 2N + 4N(N-1) \quad \text{[removing [ ] by expansion]}$$
$$= 2N + 4N^2 - 4N \quad \text{[expanding]}$$
$$\therefore 4N^2 - 2N - 306 = 0 \quad \text{[simplifying]}$$
$$2N^2 - N - 153 = 0 \quad \text{[dividing by 2]}$$
$$\therefore (2N + 17)(N - 9) = 0 \quad \text{[factorising]}$$
$$\therefore N = 9 \quad \text{[disregarding } N = -8\tfrac{1}{2} \text{ since N is a whole number]}$$

Note: The terms of the A.P. could be added until 153 is obtained - the number of terms added to get 153 is then the required answer. The nth term is $4n - 3$.

This is particularly easy using a programmable calculator. Using a Casio programmable calculator, the following program will generate displays showing the number of the term added and the sum after initially setting 0 to the memory S:

$$? \rightarrow N : S + 4N - 3 \rightarrow S$$

This is done repeatedly by pressing the (EXE) key and inputting 1, 2, 3, . . . in that order.

Exercise 17g

1. What is the sum of the first 20 terms of the sequence 2, 5, 8 ...?
2. What is the sum of the first 200 terms of the sequence 3, 6, 9?
3. An A.P. has first term of 7 and difference of 5. Find the sum of the first 50 terms.
4. An A.P. has first term of 4 and difference of -2.5. Find the sum of the first 30 terms.
5. An A.P. has 3rd term of 45 and difference of 1.1. Find the sum of the first 10 terms.
6. An A.P. has 5th term of 56 and difference of -1.1. Find the sum of the first 20 terms.
7. An A.P. has $A_1 = 6$, $A_4 = 21$. Find the sum of the first 24 terms.
8. An A.P. has $B_3 = 14$ and $B_{11} = 34$. Find the sum of the first 90 terms.
9. Find $\sum_{n=1}^{50} (2 + 3n)$
10. Find $\sum_{n=1}^{60} (4 - 7n)$
11. Find $\sum_{n=10}^{60} (3n + 1)$
12. Find $\sum_{n=15}^{35} (2n + 3)$
13. The first term of a finite A.P. is 20 and the last is 50. The sum of this series is 385. How many terms does it have?
14. An A.P. has first term 8 and difference of 3. How many terms must be added in order that the sum exceeds 65?
15. An A.P. has first term 5. The sum of its first 5 terms is 45. Find the common difference.
16. A very successful sales representative makes 35 sales in the first month of employment and for the next two years increases the number of sales each month by 4. Over the two year period how many sales were made?
17. After a scandal involving senior members of the ruling party the number of people who resign from the party each week increases by 26, on average, per week. The first week after the scandal first broke there were 134 resignations. This pattern continued for three and a half months. How many people resigned during that period?
18. A runner on a special programme seeks to increase the number of kilometres run by 12 each week. During the fourth week 61 kilometres are run. This programme is followed for 12 weeks. What is the total distance run during the period?

19. Dexter, weighing 136kg, goes on a special diet for a period of 16 weeks. The amount lost per week decreases by 0.36kg. During the fifth week 5.42kg were lost. What was Dexter's weight at the end of the diet?
20. People start arriving at a hotel featuring a well-known entertainer. The doors are opened at 6.00pm. There is seating for 353 people. The number of people arriving increases by 13 every five minutes. During the first 10 minutes 23 people arrived. After what time will people have to stand?

Geometric Sequences

A **geometric sequence** or **geometric progression** (G.P.) is a sequence where each term after the first gives a constant value when *divided* by the previous term. The value obtained when dividing is called the **ratio** of the G.P., or the **common ratio** of the G.P.

Example O: The terms of the sequence $-2, 1, \frac{-1}{2}, \frac{1}{4}, \ldots$ are in geometric sequence because each term, when divided by the previous term, is equal to $\frac{-1}{2}$ as the following shows: $1 \div -2 = \frac{-1}{2}$, $\frac{-1}{2} \div 1 = \frac{-1}{2}$, $\frac{1}{4} \div \frac{-1}{2} = \frac{-1}{2}$

The nth term of the geometric sequence with first term A_1 and ratio R is:

$$\boxed{A_n = A_1R^{n-1}}$$

This follows from the enumerated sequence: $A_1, A_1R, A_1R^2, A_1R^3, \ldots$.
The power of R is always *one less* than the number of the term.

Example P: Inflation has been running at 14% for a number of years in a certain country. After one year of inflation a pair of shoes cost $30.00.
a.Find the price of this pair of shoes after 2 and after 3 years.
b.Show that these prices are in geometric progression.
c.Find the common ratio and hence the price after *n* years.
d.Find how many full years will elapse before the cost of a pair of shoes will exceed $1 000.00.

Solution:

a. After one year the price is $30.00.
After two years the price will be $30.00 + 14% of $30.00 = $34.20.
After three years the price is $34.20 + 14% of $34.20 = $38.99.

b. $\frac{38.99}{34.20} = 1.14$ and $\frac{34.20}{30} = 1.14$, thus the prices are in geometric progression.

c. R = 1.14 . So after *n* years the price will be $30(1.14)^{n-1}$.

d. Letting t be the number of years which have elapsed when the price reaches $1 000.00, then:

$30(1.14)^{t-1} = 1\,000$ [since $A_n = A_1R^{n-1}$]

$\therefore (1.14)^{t-1} = \frac{100}{3}$ [dividing by 30]

$\therefore \ln(1.14^{t-1}) = \ln(\frac{100}{3})$ [taking logs]

$\therefore (t-1)\ln(1.14) = \ln(\frac{100}{3})$ [log A^n = n log A]

$\therefore t - 1 = \frac{\ln\left(\frac{100}{3}\right)}{\ln(1.14)}$ [dividing by $\ln(1.14)$]

$\therefore t = \frac{\ln\left(\frac{100}{3}\right)}{\ln(1.14)} + 1$ [adding 1]

$= 27.76$ years

$\therefore$ after 28 full years the price will exceed $1 000.00

Note: Using the formula, price = $30(1.14)^{n-1}$, a trial and error method would solve the problem also.

Exercise 17h

1. The following sequences are G.P.s; find the indicated terms and the nth term:
   a. $4, 2, 1, \ldots\ T_6, T_{10}, T_n$
   b. $8, 24, 72, \ldots\ T_6, T_9, T_n$
   c. $-1, 2, -4, 8, \ldots\ T_{10}, T_{13}, T_n$
   d. $3, \frac{-2}{3}, \frac{4}{27}, \ldots\ A_4, A_8, A_n$
   e. $\frac{3}{4}, 3\frac{3}{4}, 18\frac{3}{4}, \ldots\ A_4, A_6, A_n$
2. The following questions refer to G.P.s:
   a. $A_3 = 5, A_4 = 15$. Write down the first 6 terms and give the nth term.
   b. $A_2 = 6, A_3 = 12$. Write down the first 6 terms and the nth term.
   c. $A_5 = 6, A_6 = 9$. Find A_2, A_8 and A_n.
   d. $B_4 = 2, B_5 = 3\frac{1}{3}$. Find B_1 and the nth term.
   e. $A_1 = 4, A_3 = \frac{4}{9}$. Find A_7 and A_n.
   f. $T_4 = 11, T_8 = 176$. Find T_1 and T_n.
   g. $C_4 = -3, C_7 = 0.375$. Find C_2 and C_n.
   h. $A_1 = 50, A_2 = 75$. How many terms of this G.P. are less than 500?
   i. $A_4 = 12, A_6 = 21\frac{1}{3}$. What is the last term less than 50?
   j. $B_5 = 7$ and the ratio is 1.5. What term is 35.4375?
   k. $T_4 = 8, T_7 = 125$. What term is 1 953.125?
   l. A G.P. has first term 6 and fourth term 279.936. Find the common ratio.
3. Money placed in an investment fund increases in value by 11% a year. Initially $5000 was invested.
   a. Write out the value of the investment at the beginning of each of the first 5 years.
   b. How long will it take for this investment to reach or exceed $1 000 000 (give answer to the nearest year)?
4. The population in a country increases on average 1.3% a year. In 1991 the population is 2 150 000. Find what the population will be in 2021 if this rate of growth is maintained.

5. An accountant depreciates the value of a computer by 23% per year. The computer originally cost $10 000.
   a. If the company owning the computer sold it after 10 years what would its value be in the records of the accountant?
   b. If the company had a policy of selling computers once their depreciated value was less than a quarter of the cost price, how many years would the company retain the computer before selling it?

Sum of the First *n* Terms of a G.P.

The sum to n terms of a G.P. with first term A_1 and ratio R is:

$$S_n = \frac{A_1(R^n - 1)}{R - 1}$$

Note: If $-1 < R < 1$, the alternative form, $S_n = \frac{A_1(1 - R^n)}{1 - R}$ can be easier to use.

Example Q: In 1986 the number of tourists going to a city was 25 000. The number has been increasing by 16% every year since then. Find the total number of tourists who went to the city in the decade starting from 1986.

Solution: The number of tourists who visit the city each year is a geometric progression with $A_1 = 25\ 000$ and $R = 1.16$. The total number who have visited in ten years is:

$$S_{10} = \frac{A_1(R^{10} - 1)}{R - 1}$$

$$= \frac{25\ 000(1.16^{10} - 1)}{1.16 - 1} \quad \text{[substituting]}$$

$$= 533\ 037$$

Exercise 17i

1. What is the sum of the first 10 terms of the sequence 3, 6, 12, . . . ?
2. What is the sum of the first 10 terms of the sequence 1, 1.5, 2.25, 3.375, . . . ?
3. How many terms of the sequence 4, 3, $2\frac{1}{4}$, . . . can you add before the sum exceeds 12?
4. How many terms of the sequence 2, 14, 98, ... can you add before the sum exceeds 10 000?
5. A G.P. has a first term 4 and a second term 6. Find the sum of the first 10 terms.
6. A G.P. has a second term 11 and third term 5.5. Find the sum of the first 50 terms.
7. A house costs on average $15 000 in 1970 and increases in value by 10% every year. A man buys a house in 1970 then another one every year after that. How much money does he spend in total from 1970 to 1986?
8. A man on a diet loses 1.5% of his body-weight during each week.
   a. If he initially weighs 150 kg, write down his body-weight at the end of each of the first 5 weeks.
   b. How much does he lose in total during that time?

9. A weightlifter increases the total weight he lifted in the previous week by 0.95%. He keeps this up for 10 weeks. In the first week he lifts a total of 50 tonnes. How much in total does he lift during the 10 week period, to the nearest tonne?
10. A company which exports books to the U.S.A. manages to increase its exports by 5% each year. During the first year it sells 1 500 books. How many books does it export in its first 12 years of operation?
11. Find $\sum_{n=1}^{50} 3(1.1)^n$
12. Find $\sum_{n=4}^{40} \frac{1}{2}(1.2)^n$
13. A G.P. with first term 2 and ratio 3 has its terms added. How many terms are added to get a sum of 177 146?
14. A G.P. with first term 6 and ratio 4 is summed. How many terms are added to get a sum of 2 097 150?

Sum to Infinity of a G.P.

If successive terms of the A.P. 1, 3, 5, 7, ... are added together, the sum gets larger and larger without limit.

Similarly, with a G.P. where the ratio R *exceeds 1 in magnitude,* successive additions of the terms will yield a sum which gets bigger and bigger without limit.

Example R: Successive additions of the terms of the G.P. 1, 2, 4, 8, ... give a sum 1 + 2 + 4 + 8 + ... which tends to infinity. Thus if the sum to n terms is denoted by S_n, then $S_n \to \infty$ as $n \to \infty$.

Example S: Show that the G.P. with first term 3 and ratio 0.1 has a sum to infinity and find its value.

Solution:

$$S_n = \frac{3[1 - (0.1)^n]}{1 - 0.1} \quad \left[\text{using } S_n = \frac{A_1(1 - R^n)}{1 - R} \text{ since } -1 < R < 1\right]$$

$$= \frac{3[1 - (0.1)^n]}{\frac{9}{10}} \quad \text{[rearranging]}$$

$$= \frac{10}{3}[1 - (0.1)^n] \quad \text{[rearranging]}$$

As n increases $(0.1)^n$ decreases to zero and so S_n gets closer and closer to $\frac{10}{3}$.

The G.P. has a *sum to infinity* of $\frac{10}{3}$ which is written $S_n \to \frac{10}{3}$ as $n \to \infty$ or $S_\infty = \frac{10}{3}$.

Generally:

If $<A_n>$ is a geometric sequence with $-1 < R < 1$ then the sequence has a sum to infinity of: $\frac{A_1}{1 - R}$.

The sum to infinity of $<A_n>$ is denoted by either S_∞ or $\sum_{n=1}^{\infty} A_n$.

Example T: The sequence 3, -1, $\frac{1}{3}$, $\frac{-1}{9}$, ... is a G.P. with ratio $\frac{-1}{3}$.
Since $-1 < \frac{-1}{3} < 1$ the sum to infinity exists and has the value:

$$S_\infty = \frac{A_1}{1 - R}$$

$$= \frac{3}{1 - \frac{-1}{3}} \quad \text{[substituting } A_1 = 3, R = -\tfrac{1}{3}\text{]}$$

$$= 2.25$$

Exercise 17j

1. Which of the following G.P.s have a sum to infinity?
   a. 2, 3, 4.5, ... b. 2, 1, 0.5, 0.25, ... c. 4, 3, 2.25, ...
   d. 2.5, 3.5, 4.9, ... e. 4.9, 3.5, 2.5, ...
2. Find the following sums to infinity.
   a. $\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n$ b. $\sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^n$ c. 3 + 0.9 + 0.27 + ...
   d. -4 + 1.5 + -0.5625 + ... e. $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \ldots$
3. What is the sum to infinity of the G.P.s with:
   a. first term of 5, ratio of 0.6 b. first term of 7, ratio of 0.35
   c. 3rd term of 6, ratio of $\frac{2}{3}$ d. 4th term of 18, ratio of 0.72
   e. first term of 8, 4th term of 3.375
4. A geometric progression with first term 5 and a sum to infinity of 8 has an unknown ratio. Find that ratio.
5. Each year a country exports two thirds the quantity of the previous year's butter exports. In 1970 it exported 25 000 tonnes. If it continued to export in this way, what would be the total weight of its butter exports over all the years from 1970 onwards?
6. A man employs many people. On his birthday he decides to give each a gift of money. He gets one of his executives to put the employees in alphabetical order. The first in line he gives $1 000, then the next gets $750, the next $562.50, and so on. If it were possible to pay in this way, what would be the total value of his gifts?
7. A company hires a new accountant who is able to reduce the company's unnecessary expenditure by about 50% every year. When she was first hired, such expenditure was $813 000.
   a. Write down the unnecessary expenditure for the first 3 years of her employment.
   b. After hiring the accountant, how much can the company expect to spend unnecessarily in the future?

8. A fishing village acquires some new boats and in the first year of use they catch 19 683 tonnes of fish. The next year they catch only one third of this amount. In each following year they only catch one third of what they caught the year before. This pattern continues.
   a. Find the amount of fish they catch each year for the first 5 years.
   b. Calculate the total tonnage caught if they were to go on fishing like this until there were no more fish to catch.
9. The birth rate of a country is falling by 5% per year. In 1961 there were 65 000 births. Find how many babies will be born in the country from the beginning of 1961 if this trend continues.
10. When Kate rinses her hair after washing it with a shampoo (or a liquid soap) she removes a quantity of the shampoo. The first rinse removes 5g of shampoo. Each successive rinse removes 20% of the amount removed by the previous rinse.
    a. How much shampoo is removed at the second rinse?
    b. Write down the sequence of amounts of shampoo removed at the first 4 rinses.
    c. The sequence in (b) is a geometric sequence. Explain why.
    d. Write down an expression, in terms of n, for the amount of shampoo, A(n), removed in the nth rinse.
    e. How much shampoo was present in Kate's hair at the beginning of the wash?

Problems and Investigations

1. Sharon is going to invest $1500. Her bank offers her 4 plans: 11.8% per year, 5.8% per six months, 3.8% per four months or 0.9% per month. She intends to withdraw her money after one year. What advice would you give her?

2. The number of people per year investing in a plan increases by 50 each year. The amount they each initially invest increases by 2% per year. The plan started in 1990 with 375 people each investing $150. Any money invested increases in value by 11% a year. Investigate this in order to determine when the total value of money invested exceeds $1 000 000.

3. Patterns occur in nature, eg the spiral growth of plants, and in the designs of many cultures, eg Mesa Verde pottery (USA), Maori Carvings (NZ) and the Begho smoking pipes (Ghana, Africa). Investigate some geometric patterns of these types and present a mathematical analysis of them.

18. GRADIENT FUNCTION

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 MEASUREMENT AND CALCULUS

- *to sketch the graph of a gradient function from a graph of a function and explain the relationship between them*

  (Suggested learning experience: students should be investigating and interpreting the relationship between a graph and its slope from a rate of change point of view, for example, by using a graphics calculator or suitable graphics package to "zoom in" on a point of a graph until the graph appears to be a straight line, and then find its gradient)

Introduction

In chapter 10 straight lines were studied, including an important feature of them, their **gradient** or **slope**.

This was used in problems where there was a constant **rate of change**. In such cases it was found that the graph obtained was a straight line and the gradient was the rate of change.

Example A:

A tank initially contains 50L of water. A tap delivers water at the rate of 4.5L per minute.

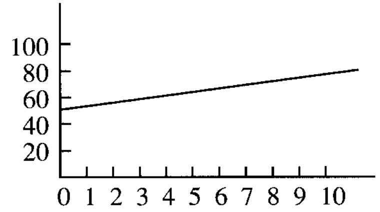
i. Draw a graph of volume against time.
ii. Find the equation of the function whose graph was drawn.

Solution

i. The following table of values gives the volume in the tank for selected times.

Time (minutes)	0	1	2	5	10
Volume (litres)	50	54.5	59	72.5	95

The graph of this relationship is shown below:



ii. The graph is a straight line with gradient 4.5 and intercept on the vertical axis of 50 ∴ letting V represent volume in litres and t time in minutes.

$$V = 4.5t + 50$$

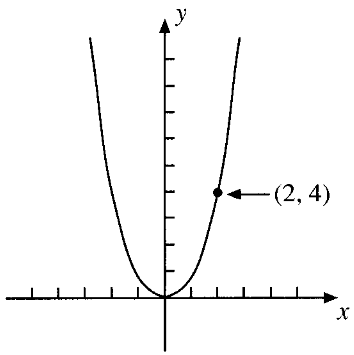
Note: the **gradient** (4.5) of this graph is the **rate of change** of the variable V with respect to time.

Rate of Change of any Variable

In most situations the graph of one variable against the other is not a straight line. It will be found in these cases that the rate of change is not constant but changes.

Example B:

Find the rate of change of y with respect to x on the graph $y = x^2$ when $x = 2$.



The graph is that of $y = x^2$.

If we got a magnifying glass and examined the graph of $y = x^2$ at the point (2, 4) with a sufficiently powerful magnification the graph would appear straight at that point.

The **gradient** or **slope** of the graph at the point (2, 4) is the gradient of this straight line. This is also the **rate of change of *y* with respect to *x*** when $x = 2$.

If the student lacks a graphical calculator or other device which enables the graph to be 'blown up' at the point (2, 4) so as to appear straight then the slope may be found using the technique given below.

i. select a pair of values very close to the x value in question, in this case 2, one above and one below, e.g. $x = 1.99$ and $x = 2.01$.

ii. calculate the corresponding y values.

x	1.99	2.01
y	3.9601	4.0401

iii. calculate the slope using $\frac{y_2 - y_1}{x_2 - x_1} = \frac{4.0401 - 3.9601}{2.01 - 1.99}$

$= 4$

Example C:

For the function $y = 2x - 4$ calculate:

i. The gradient when $x = 1$.
ii. The slope when $x = 3$.
iii. The rate of change of y with respect to x when $x = 2$.

Solution:

The student should recognise that the equation $y = 2x - 4$ is that of a straight line having a constant gradient or slope of 2.

The rate of change of y with respect to x is the same as the gradient and will also equal 2.
Thus the answer to all three questions is 2.
[These values can also be found by calculation as in example B.]

Use of Graphical Calculators

In Example B reference was made to the facility on calculators such as the Casio *fx*-7700 GB, Hewlett Packard 48G and others with graphical capabilities to 'blow up' sections of graphs.
This is done by using the zoom function which enables the user to 'cut out' a small snip of the graph and blow it up so that it fills the screen. Use of the TRACE function then enables the user to write down the co-ordinates of points anywhere on the graph.

Exercise 18a

1. For the graph of $y = x^2$ calculate the gradient when:
   a. $x = 1$ b. $x = -3$

2. For the graph of $y = x^2$ calculate the slope when:
   a. $x = 0$ b. $x = -2$

3. For the graph of $y = x^2$ calculate the rate of change of y with respect to x when:
   a. $x = 2$ b. $x = -1$

4. For the graph of $y = 3x - 1$ calculate:
   a. The gradient when $x = 2$.
   b. The slope when $x = 4$.
   c. The rate of change of y with respect to x when $x = 5$.

5. For the graph of $y = 4 - 2x$ calculate:
   a. The gradient when $x = 1$.
   b. The slope when $x = 3$.
   c. The rate of change of y with respect to x when $x = -2$.

6. For the graph of $y = 4$ calculate:
   a. The gradient when $x = 2$.
   b. The slope when $x = -2$.
   c. The rate of change of y with respect to x when $x = -1$.

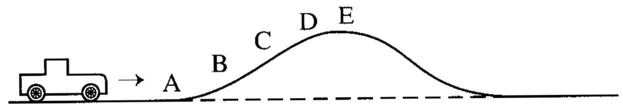
7. For the graph of $y = x^3$ calculate:
   a. The slope when $x = 2$.
   b. The gradient when $x = 3$.
   c. The rate of change of y with respect to x when $x = -1$.

8. For the graph of $y = 2x^2 + x$ calculate:
   a. The slope when $x = 1$.
   b. The gradient when $x = 2$.
   c. The rate of change of y with respect to x when $x = -3$.

The Practical Effects of Variable Gradients

We all encounter situations involving variable gradients.

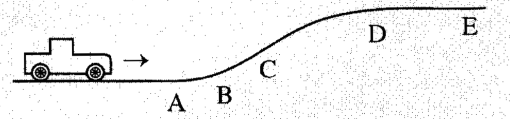
Example D: The driver of a car encounters the hill as shown below:



At point A the driver, going at a *constant* speed and never accelerating, is on the 'flat'. As the car moves from A to B to C the hill gets 'steeper and steeper'; without acceleration the driver will notice an increasing slowing effect. 'Steepness' is a term used by people in everyday life to describe **gradient**. The greater the gradient, the greater the 'steepness'. As the driver proceeds from C to D to E the hill becomes less and less steep. The gradient is decreasing and the 'slowing' effect will decrease. Once the car passes the top of the hill at E it will start to speed up again as it is going down the hill now. In mathematical terms C, the point of greatest steepness, is called a 'point of inflection'.

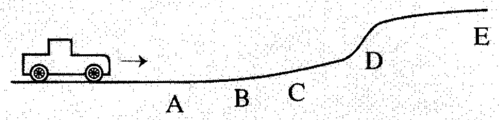
Exercise 18b

1. a. Describe the passage of a car which is going at a constant speed as it begins to travel up the hill. Throughout the journey the driver never accelerates.



   b. Which point is the 'point of inflection'?

2. Repeat question 1 for the hill below.



3. Draw a graph which would describe the following trip.

   I am proceeding along a flat road. I notice a slight slope at A which increases as I pass from A to B to C. After I pass through C the hill I have come up starts to flatten out as I pass through D and the road after E is once again flat.

4. Draw a graph which describes this child's path down an amusement park slide.

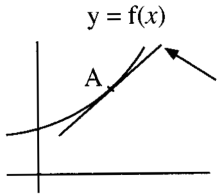
   It started gently but quickly got steeper. Then the slide seemed to be getting flatter and flatter till it was actually flat then it suddenly got steeper and steeper again till I landed on the water below.

5. Draw a graph which might describe this company's profits.

   We started out making a profit of about $1 000 a day in January This slowly increased over 6 months when we were making $1 500 a day. Our profits rapidly increased till the end of August. After that the rate of increase of profit decreased till from the beginning of December till the end of the month profit was constant at $3 000 a day.

Gradient Function

The function which arises from the calculation of the gradient, slope or rate of change at every point of a function's graph is called the **gradient function** for that particular graph.



the gradient of this tangent at the point A is the gradient of the function f(x) at point A.

At points on the graph where there is a tangent the gradient function gives the slope of the tangent to the curve at that point. See diagram.

Example E:
Find a formula for the gradient function of the graph $y = x^2$.

Solution:
Using a graphical calculator or the method of example B the student will obtain the following values:

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9
gradient	-6	-4	-2	0	2	4	6

Quite clearly the gradient function for $y = x^2$ is the function $y = 2x$.

Example F:
Find a formula for the gradient function of $y = 3x + 1$.

Solution:
$y = 3x + 1$ represents a straight line and has a constant gradient of 3 thus its gradient function has the formula $y = 3$.

[This could also be obtained by using the method in example E.]

Exercise 18c

1. For the function $y = x^3$:
   a. Copy and fill in this table of values.

x	-3	-2	-1	0	1	2	3
y							
gradient							

   b. Obtain a formula for the gradient function.

2. For the function $y = x^4$:
   a. Copy and fill in a table of values similar to that in question 1.
   b. Find the formula for the gradient function.

3. Repeat the above and find formulae for the gradient functions of the following:
   a. $y = x^5$ b. $y = 2x^2$ c. $y = x^2 - x$
   d. $y = (x + 2)^2$ e. $y = 2x + 1$ f. $y = 7$

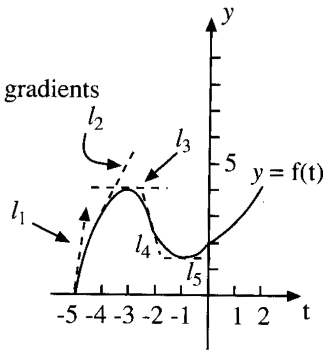
Sketching Graphs of Gradient Functions from the Graphs of Functions

Example G:

a. Sketch the graph of the gradient function for the function $y = f(t)$ whose graph is shown in the diagram. y is measured in thousands and t in months.

b. If t represents time measured from January 1 and f(t) represents the number of immigrants arriving in a city, interpret the gradient function.

Solution:



Examination of this graph shows that from -5 to -3 the gradient decreases. The

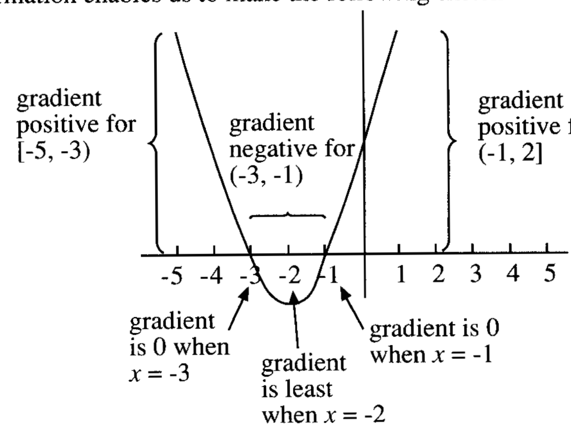
of the graph at the points (-5, 0), (-4, 3), (-3, 4) are the gradients of the lines l_1, l_2, l_3, which are the tangents at those points.

These gradients are decreasing because the tangents are becoming less steep.

Once the graph has passed through (-3, 4) the gradients become negative, reaching their lowest value at the point (-2, 2.5) as shown by the line l_4 which is not a tangent but whose slope has the same value as the gradient function at that point. (-2, 2.5) is a **point of inflection**.

The gradients then gradually increase taking the value 0 at the point (-1, 1.7), and then increase from there on.

This information enables us to make the following sketch of the gradient function:



Note: The **interval notation** [-5, -3) means $-5 \leq x < -3$; (-3, -1) means $-3 < x < 1$; (-1, 2] means $-1 < x \leq 2$. (A curved bracket excludes an endpoint, a square bracket includes an endpoint.)

b. From five to three months before January 1 there was an increasing number of immigrants arriving in the city. The rate of increase steadily decreased until October when the actual number of immigrants began to decrease reaching its greatest rate of decrease near the beginning of November. The number continued to decrease until the beginning of December, when the number of immigants began increasing again with the rate of increase steadily increasing until at least February 1.

Note: If you initially find it hard to sketch freehand the graph of a gradient function then you can get a good sketch as follows. Make a large neat copy of the graph given by plotting points on fine mesh graph paper. Calculate approximate values for the gradients at those points, then plot those values on another graph.

Example H:

For the graph in example G, after making your large copy on fine mesh paper a good sketch of the gradient function could be made by calculating the gradients at -5, -4.5, -4, -3.5, -3, -2.5, -2, . . . , 2, 2.5, 3 then plotting them.

A less accurate but still reasonable sketch could be obtained by calculating the gradients at -5, -4, -3, -2, -1, 0, 1, 2, 3.

Calculation of the gradient at any point is by use of the formula (see next example).

Example I:
Calculate the gradient at the point (-4, 3) for $y = f(t)$ of example G.

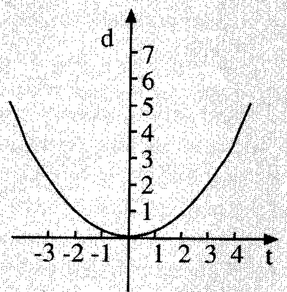
Solution:
Let (-3.9, f(-3.9)), (-4.1, f(-4.1)) be the points corresponding to -3.9 and -4.1 which you can find by checking your graph drawn on fine mesh paper.

The gradient at (-4, 3) can be estimated to be $\frac{f(-3.9) - f(-4.1)}{-3.9 - -4.1}$

[using the formula gradient $\frac{y_2 - y_1}{x_2 - x_1}$].

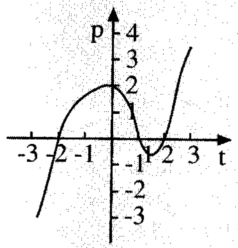
Exercise 18d

1. a. Sketch the gradient function for the graph shown.



b. If the graph represents the distance from a house in km with t being time measured in hours, interpret the gradient funciton.

2. a. Sketch the gradient function for the graph shown.



b. The graph represents the profit p of a business in units of $1 000 per day and t represents time in months measured from March 1. Interpret the gradient function.

3. a. Sketch the graph going through these points:
(-3, 0), (-2, 5), (-1, 8), (0, 9), (1, 8), (2, 5), (3, 0)
b. Sketch the gradient function for the graph drawn.

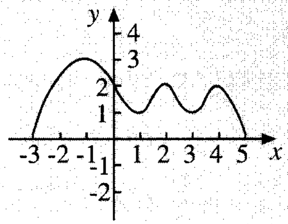
4. a. Sketch the graph going through these points:
(-2, -6), (-1, 0), (-0.5, 0.4), (0, 0), (0.5, -0.4), (1, 0), (2, 6)
b. Sketch the gradient function of the above graph.

5. a. Sketch the graph going through the following points:
(-2, 0.25), (-1, 0.5), (0, 1), (1, 2), (2, 4), (3, 8)
b. Sketch the gradient function.

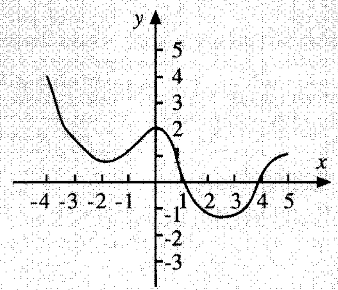
6. a. Plot these points and join them with straight line segments:
(-4, 2), (-3, 3), (-2, 4), (-1, 3), (0, 0), (1, -1), (2, -2), (3, -1), (4, 0)
b. Sketch the gradient function for the graph drawn in a.

7. a. Plot these points and draw a neat sketch of the graph going through them:
(-4, 0), (-3.5, 2), (-3, 3), (-2.5, 3.5), (-2, 3), (-1, 1), (-0.5, 0.5), (0, 1), (1, 1.5), (2, 1.8), (3, 1.9), (4, 4).
b. Sketch the gradient function for the graph drawn in 7a above.

8. Sketch the gradient function for the function whose graph is shown.

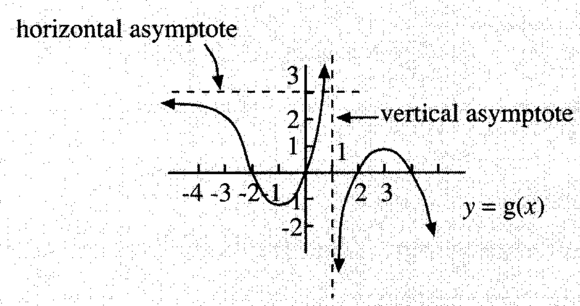


9. Sketch the gradient function of the graph shown below.



Problems and Investigations

1.



a. Examine the above graph and describe a real situation which might be modelled by the graph.
b. Sketch the gradient function of the graph.

2. Investigate the gradient functions of each of the following with the aim of finding formulae for them.

a. $y = \frac{1}{x}$ b. $y = \sqrt{x}$ c. $y = e^x$ d. $y = ln x$ e. $y = \log x$

f. $y = \frac{1}{x+2}$ g. $y = |x|$ h. $y = x + \frac{1}{x}$ i. $y = x ln x$ j. $y = xe^x$

19. DERIVATIVES

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 MEASUREMENT AND CALCULUS

- *to determine an expression for the rate of change of a variable*

Introduction

Derivatives are a part of mathematics called **calculus.**

The derivative of a function f at point x in the function's domain is the gradient function's value at x.

Example A:
It was shown in example E of the previous chapter that the gradient function of $f(x) = x^2$ is the function having the expression $y = 2x$.

By using identical methods it can be shown that the derivative or gradient function of x^3 is $3x^2$ and of x^4 is $4x^3$.

Generally:

The derivative of x^n is nx^{n-1}

Example B: The derivative of x^4 is $4x^3$ and the derivative of x^{43} is $43x^{42}$.

Note:

a. The derivative of x is 1.

Proof: x can be written as x^1. The derivative of x^1 is $1x^0 = 1$.

b. The derivative of 1 is 0.

Proof: 1 can be written as x^0. The derivative of x^0 is $0x^{-1} = 0$.

Differentiation

The process of *finding a derivative* is called **differentiating**. Functions which *can be differentiated* are **differentiable**. Substitution is used to find the value of the derivative for a particular value in the domain.

Example C: Differentiate x^7 and find the value of the derivative when $x = 2$.

Solution: Differentiating x^7 gives the derivative $7x^6$.
When $x = 2$, the value of the derivative is $7 \times 2^6 = 448$. [substituting $x = 2$]

$f'(x)$, $\frac{dy}{dx}$ or y' all mean the derivative of the function, $y = f(x)$. $f'(a)$ means the derivative of the function $y = f(x)$ at $x = a$.

Example D:

a. $f(x) = x^{24}$ $\therefore f'(x) = 24x^{23}$

b. $y = x^{11}$ $\therefore \frac{dy}{dx} = 11x^{10}$

c. $u = r^{13}$ $\therefore \frac{du}{dr} = 13r^{12}$

d. $g = h^9$ $\therefore g' = 9h^8$

When a term has a coefficient a, the term is differentiated using:

$(ax^n)' = anx^{n-1}$

Example E:

a. $f(x) = 5x^3$
$\therefore f'(x) = 5 \times 3 \times x^2$
$= 15x^2$

b. $g(x) = 6x^4$
$\therefore g'(x) = 6 \times 4 \times x^3$
$= 24x^3$

If an expression contains several terms, the expression is differentiated by differentiating each term separately. Brackets may need to be removed first.

Example F: Differentiate: a. $y = x^{10} + x^9 - x^7 + 2x$ b. $u = 8t^6 + 7t^2 + 5$

Solution: a. $\frac{dy}{dx} = 10x^9 + 9x^8 - 7x^6 + 2$

b. $\frac{du}{dt} = 48t^5 + 14t + 0$
$\therefore \frac{du}{dt} = 48t^5 + 14t$

Example G : Differentiate: $(x^2 + 1)^2$

Solution: $(x^2 + 1)^2 = x^4 + 2x^2 + 1$ [expanding]
$\therefore f'(x) = 4x^3 + 4x$ [differentiating each term]

Good setting out helps to avoid making mistakes. Always take care *not* to write incorrect mathematical statements.

Example H: Differentiate x^7.

Solution: $f(x) = x^7$
$\therefore f'(x) = 7x^6$

Note: Writing $x^7 \to 7x^6$ is acceptable but $x^7 = 7x^6$ is not, because $x^7 \neq 7x^6$.

Exercise 19a

1. Differentiate:

a. x^5	b. x^{22}	c. x^{48}	d. $3x^7$
e. $5x^9$	f. $10x^2$	g. $\frac{x^2}{3}$	h. $8x^3 - 3x^2 + 6$
i. $\frac{x^6}{2}$	j. $\frac{4x^2}{3}$	k. 9	l. $5x + 7$

m. $2x^2 + 3x$ n. $7x^{10} - \frac{8x^{12}}{5}$ o. $(x + 1)^2$
p. $(2x + 1)^2$ q. $(3x + 4)$ r. $(x^2 + x)^2$
s. $(2x - 3)^2$ t. $x(x + 3)$ u. $(x + 1)(x + 4)$
v. $(x^2 + 1)(x + 3)$ w. $x^3(x^2 + 3x + 2)$
x. $(x^2 + 1)(x^2 + 2x + 2)$ y. $x(x + 2) + (x + 1)(x + 3)$
z. $x^2(x^2 - 3) + \frac{1}{2}(x^2 + 3x)$

2. a. $y = 4x^3$. Find $\frac{dy}{dx}$ b. $y = 6x^2$. Find $\frac{dy}{dx}$
c. $f(x) = 5x^2 + 3x - 2$. Find $f'(x)$.
d. $h(t) = 3t^3 + 4t^2 - 3t + 7$. Find $h'(t)$.
e. $g(u) = 2(u^2 + 3u) + 3(u + 4)$. Find $g'(u)$.
f. $w = 3x^2 + 2t$. Differentiate w with respect to x where t is a constant.
g. $w = 3x^2 + 2t$. Differentiate w with respect to t where x is a constant.
h. $y = 4x^5 + \frac{x}{7}$. Find y'.
i. $p = 2r(r + 1)(r + 3)$. Find $\frac{dp}{dr}$.
j. $u = (2t + \frac{3}{t})(3t^2 + 4t)$. Find u'.

3. a. $f(x) = x^2 + 3x$
i. Find $f'(2)$. ii. Find $f'(3.5)$.
iii. Solve the equation $f'(x) = 4$.
b. $g(x) = x^2 - 2x + 3$
i. Find $g(2)$. ii. Find $g'(2)$.
iii. Find $g'(3)$. iv. Solve $g'(x) = 5$.
c. $h(x) = 2x^3 - 3x^2 + 5x - 11$
i. Find the derivative of h when x is 5.
ii. Find $h'(x)$ when x is 2.
iii. Solve $h'(x) = 5$.
d. $y = 3(x + 1)^2$
i. Find y' when $x = -2$. ii. Find $\frac{dy}{dx}$ when $x = \frac{1}{2}$.
iii. Evaluate $\left.\frac{dy}{dx}\right|_{-1}$
This means 'find the value of $\frac{dy}{dx}$ when $x = -1$'.
iv. Solve $\frac{dy}{dx} = 5$.

e. $y = \frac{1}{3}x^3 - x^2 - 8x + 2$
i. What is y when $x = 2$? ii. Find y' when $x = 1$.
iii. Solve $\frac{dy}{dx} = 0$. iv. Solve $y' = 7$.

Negative Indices

Differentiation of expressions involving negative indices is dealt with in the same way as for those with positive indices, taking care with the sign.

Example I: Differentiate the following:

a. $f(x) = x^{-2}$ b. $\frac{1}{x^3}$ c. $\frac{4}{x^5}$ d. $\frac{3}{5x^6}$

Solution:

a. $f(x) = x^{-2}$
$f'(x) = -2x^{-3}$ (or $\frac{-2}{x^3}$)

b. $\frac{1}{x^3} = x^{-3}$
$\left(\frac{1}{x^3}\right)' = -3x^{-4}$ (or $\frac{-3}{x^4}$)

c. $f(x) = \frac{4}{x^5}$
$= 4x^{-5}$
$f'(x) = -20x^{-6}$ (or $\frac{-20}{x^6}$)

d. $\frac{3}{5x^6} = \frac{3}{5}x^{-6}$
$\left(\frac{3}{5}x^{-6}\right)' = \frac{-18}{5}x^{-7}$ (or $\frac{-18}{5x^7}$)

Exercise 19b

Differentiate each of the following:

1. x^{-4}	2. $\frac{1}{x^5}$	3. $\frac{1}{u^7}$	4. $\frac{3}{u^4}$	5. $\frac{5}{x^6}$
6. $\frac{2}{3y^5}$	7. $\frac{3}{4x^2}$	8. $\frac{1}{5x^4}$	9. $\frac{3}{7x^2}$	10. $\frac{2}{x^3} + \frac{3}{x^4}$

Rational Powers

Expressions involving rational powers are dealt with using the methods in Chapter 4.

Example J: Differentiate the following:

a. $(x)^{\frac{2}{3}}$ b. $\sqrt[3]{x}$ c. $\sqrt[5]{u^3}$

Solution:

a. $\left[(x)^{\frac{2}{3}}\right]' = \frac{2}{3}(x)^{\frac{-1}{3}}$ or $\frac{2}{3(x)^{\frac{1}{3}}}$ or $\frac{2}{3\sqrt[3]{x}}$

b. $\sqrt[3]{x} = (x)^{\frac{1}{3}}$
$\left[(x)^{\frac{1}{3}}\right]' = \frac{1}{3}(x)^{\frac{-2}{3}}$ (or $\frac{1}{3(x)^{\frac{2}{3}}}$ or $\frac{1}{3\sqrt[3]{x^2}}$)

c. $\sqrt[5]{u^3} = (u)^{\frac{3}{5}}$

$$\left[(u)^{\frac{3}{5}}\right]' = \frac{3}{5}(u)^{\frac{-2}{5}} \text{ or } \frac{3}{5(u)^{\frac{2}{5}}} \text{ or } \frac{3}{5\sqrt[5]{u^2}}$$

Use of Calculators to Find Derivatives

There are some calculators available which work out derivatives. An example is the Hewlett Packard HP 48G. This is done by means of the SYMBOLIC MENU and the DIFFERENTIATE option.

Exercise 19c

1. Differentiate each of the following:

a. $\sqrt[4]{x}$ b. $\sqrt[5]{r}$ c. $\sqrt[3]{u^2}$ d. $\sqrt[5]{t^4}$ e. $\sqrt[3]{y^5}$

f. $\frac{1}{\sqrt{x}}$ g. $\frac{1}{\sqrt{u^3}}$ h. $\frac{4}{\sqrt[4]{r^3}}$ i. $\frac{5}{\sqrt[6]{u^5}}$ j. $\frac{1}{3\sqrt{x}}$

k. $\frac{1}{5\sqrt[3]{x^2}}$ l. $\frac{2}{3\sqrt[4]{x^3}}$ m. $\frac{3}{5\sqrt[7]{x^4}}$ n. $\frac{2}{3\sqrt{x^3}}$ o. $\frac{5}{\sqrt[3]{x^7}}$

2. a. $f(x) = \sqrt{x}$ Find $f'(4)$ b. $f(t) = \frac{1}{t^3}$ Find $f'(2)$

c. $g(x) = \frac{2}{x^4}$ Find $g'(3)$ d. $h(x) = \frac{1}{2\sqrt{x}}$ Find $h'(9)$

e. $y = \sqrt[3]{x^2}$ Find $\left.\frac{dy}{dx}\right|_{27}$ f. $y = \frac{3}{x^2}$ Find $\left.\frac{dy}{dx}\right|_{3}$

g. $u = \frac{2}{3}\sqrt{r}$ Find $\left.\frac{du}{dr}\right|_{25}$

h. $s = \sqrt[3]{t}$ $u = 7\sqrt[3]{t^2}$ Which is the greater, $s'(4)$ or $u'(27)$?

i. $s = 5\sqrt[4]{x^3}$ $u = 7\sqrt[3]{x^2}$ Which is the greater, $s'(16)$ or $u'(64)$?

j. $g(t) = \sqrt{t}$ $h(t) = \frac{1}{\sqrt{t}}$ Find $g'(4)$ and $h'(4)$

Problems and Investigations

1. $f(x) = Ax^B$. Find A and B if:

a. $f(4) = 6$ and $f'(4) = \frac{3}{4}$. b. $f(8) = 4$ and $f'(8) = \frac{1}{6}$.

2. $h(x) = \frac{a}{x^2}$, $h(b) = \frac{5}{4}$ and $h'(b) = \frac{-5}{4}$. Find a and b.
3. Who was Leibnitz, and what did he discover? Explore the development of calculus, its notations and applications.

20. CALCULUS AND TANGENTS

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 MEASUREMENT AND CALCULUS

- *to sketch the graph of a gradient function*

  (Suggested learning experience: students should be investigating and interpreting the relationship between a graph and its slope)

Introduction

From chapter 18 we have the following extremely important result:

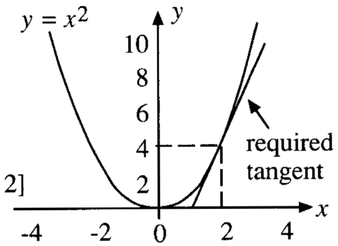
> The gradient of the tangent to the curve $y = f(x)$ at $(a, f(a))$ is $f'(a)$.

Example A: Find the equation of the tangent to $y = x^2$ when $x = 2$.

Solution: When $x = 2$, $y = 4$, [since $y = 2^2$]
thus the tangent goes through the point (2, 4).

The gradient of the tangent at x is $\frac{dy}{dx} = 2x$
[differentiating]

$\therefore$ the gradient when $x = 2$ is $2 \times 2 = 4$
[substituting $x = 2$]



The equation of the straight line is:

$y - 4 = 4(x - 2)$ [from $y - y_1 = m(x - x_1)$]

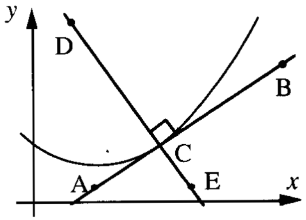
$\therefore y - 4 = 4x - 8$ [expanding]

$\therefore y = 4x - 4$ [simplifying]

Normals

At any point, the **normal** to the graph of a function is the *perpendicular line* to the tangent at that point.

In the diagram, AB is the tangent to the curve at point C, and DE is the normal at point C.



> If the gradient of the tangent at a point is m then the gradient of the normal at the same point is $\frac{-1}{m}$.

Note: The reason for this is that gradients of two perpendicular lines have a product of -1. Thus $m \times \frac{-1}{m} = -1$.

Example B: Find the equation of the normal to the graph of $y = x^3$ at the point where $x = 1$.

Solution: The gradient of the tangent is $\frac{dy}{dx} = 3x^2$.

$\therefore$ at $x = 1$, the gradient of the tangent is $3 \times 1^2 = 3$ [substituting in $3x^2$]

$\therefore$ gradient of normal is $\frac{-1}{3}$ [since $3 \times -\frac{1}{3} = -1$]

When $x = 1$, $y = 1$, so the normal passes through (1, 1) [substituting in $y = x^3$]

$\therefore$ equation of the normal is:

$y - 1 = \frac{-1}{3}(x - 1)$ [substituting into $y - y_1 = m(x - x_1)$]

$\therefore 3y - 3 = -(x - 1)$ [multiplying by 3]

$\therefore 3y - 3 = -x + 1$ [expanding]

$\therefore 3y + x - 4 = 0$ [adding x, subtracting 1]

Example C: Find the points on $y = x^2$ with:

i. a tangent with gradient 3. ii. a normal with gradient 2.

Solution:

i. Gradient of the tangent is $2x$. [differentiating x^2]

$\therefore$ point with tangent having gradient 3 satisfies $2x = 3$.

$\therefore x = 1.5$ [dividing]

at $x = 1.5$, $y = (1.5)^2$ [since $y = x^2$]

$= 2.25$

The point on $y = x^2$ where the tangent has gradient 3 is (1.5, 2.25).

ii. The gradient of the normal at (x, x^2) is $\frac{-1}{2x}$. [since $2x \times \frac{-1}{2x} = -1$]

$\therefore$ The point with a normal of gradient of 2 satisfies:

$\frac{-1}{2x} = 2$

$\therefore -1 = 4x$ [multiplying by $2x$]

$\therefore x = \frac{-1}{4}$

$\therefore y = \left(\frac{-1}{4}\right)^2$ [substituting $x = \frac{-1}{4}$ into $y = x^2$]

$= \frac{1}{16}$

$\therefore$ the point is $\left(\frac{-1}{4}, \frac{1}{16}\right)$.

Tangents and Calculators

There are some calculators available such as the Hewlett Packard HP 48G which by use of certain modes enable the user to find the slope and the equation of the tangent at any point on the graph.

Exercise 20

1. Find the gradient of the tangent to each of the following curves at the point required:
   a. $y = x^2 + x$ when $x = 2$
   b. $y = x^2 - 2x$ when $x = 1$
   c. $y = x^2 + x$ when $x = -2$
   d. $y = x^3 - x^2$ when $x = 1$
   e. $y = 2\sqrt{x}$ when $x = 4$

2. Find the equation of the tangent to each of the following curves at the point indicated:
   a. $y = x^2$ when $x = 2$
   b. $y = 2x^2$ when $x = -1$
   c. $y = \frac{x^2}{2}$ when $x = 3$
   d. $y = \frac{x^3}{2} + 1$ when $x = 1$
   e. $y = \sqrt{x}$ when $x = 4$

3. Find the equation of the normal to each of the following curves at the indicated point:
   a. $y = x^2$ when $x = 2$
   b. $y = \sqrt{x}$ when $x = 4$
   c. $y = x^2 - 3x$ when $x = 2$

4. Find the point or points on each of the following graphs where the tangents have the gradient indicated:
   a. $y = x^2$, gradient is 3
   b. $y = \frac{1}{3}x^3$, gradient is 4

5. Find the point on each of the following graphs where the normals have the given gradients:
   a. $y = x^2$, gradient is $\frac{-1}{2}$.
   b. $y = x^2 - 3$, gradient is 2
   c. $y = \sqrt{x}$, gradient is -4

Problems and Investigations

1. Find the value of k so that the tangents to the graphs of $y = \frac{k}{x}$ and $y = x^2$ intersect at right angles at the point where the two graphs cut.

2. Find the value of k so that the tangents to the graphs of $y = \frac{k}{x}$ and $y = x^3$ intersect at right angles at the point where the two graphs cut.

3. $y = x(Cx + D)$ passes through the point (-2, -4) and has a tangent with a gradient of 1 at that point. Find C and D.

21. INCREASING AND DECREASING FUNCTIONS: TURNING POINTS

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

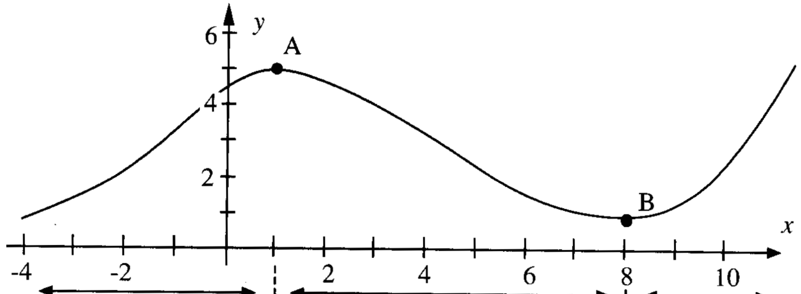
LEVEL 7 MEASUREMENT AND CALCULUS

- *to determine and use an expression for the rate of change of a variable*

Increasing and Decreasing Functions

A function is **increasing** on a set of values, if as the values increase, the function's values increase.
A function is **decreasing** on a set of values, if as the values increase, the function's values decrease.

Example A:



As the x values increase from - 4 to 1, the y values *increase* from 1 to 5. The function $y = f(x)$ is said to be *increasing* on the interval $-4 < x < 1$.

As the x values increase from 1 to 8, the y values *decrease* from 5 to 1. The function $y = f(x)$ is said to be *decreasing* on the interval $1 < x < 8$.

As the x values increase from 8 the function $y = f(x)$ is again an increasing function.

Turning Points

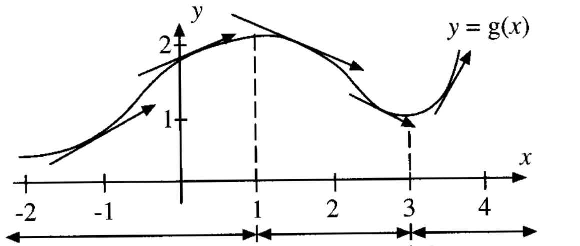
A point at which a function changes from increasing to decreasing or vice versa is called a **turning point** (abbreviation **T.P.**).

If the function changes from increasing to decreasing, the turning point is called a **local maximum**. In Example A, the point A (1, 5) is a local maximum because at the value 1 the function reaches a maximum of 5 for all values close to 1.

Note: When the x value exceeds about 11, the value of the function becomes greater than 5.

Local minimums are turning points where the function changes from being a decreasing function to being an increasing function. In Example A, the point B (8, 1) is a local minimum.

Tangents to Graphs



function increasing for $x < 1$; tangents have *positive* gradients.

function decreasing for $1 < x < 3$; tangents have *negative* gradients.

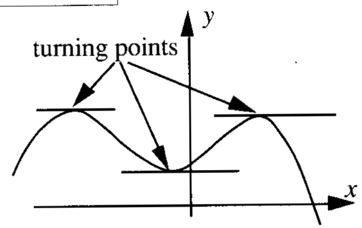
function increasing for $x > 3$; tangents have *positive* gradients.

Any tangent drawn at a point on the graph of a function where the function is increasing is 'tilted up' and hence has a *positive* **gradient.** Thus at points on the graph of a function which is increasing, the derivative will be positive. Similarly when the function is decreasing the derivative is negative. This can be written:

> If $y = f(x)$ is an increasing function then $f'(x) > 0$.
> Similarly, if the function is decreasing then $f'(x) < 0$.

At turning points, tangents are horizontal and have a gradient of 0.

> At turning points, $f'(x) = 0$.



Determining the Nature of Turning Points

A graph will show clearly whether a turning point is a local maximum or minimum.

Example B: Find the turning point(s) of $y = 2x^2 - 8x + 4$ and determine their nature.

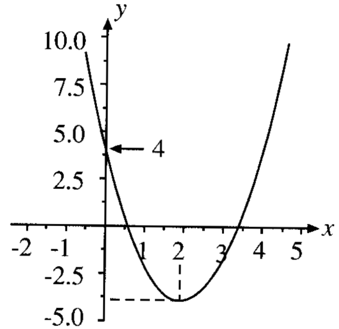
Solution:

$y' = 4x - 8$	[differentiating $y = 2x^2 - 8x + 4$]
$4x - 8 = 0$	($y' = 0$ at the turning point(s))
$\therefore\ x = 2$	[solving for x]
At $x = 2$, $y = 2 \times 2^2 - 8 \times 2 + 4$	[substituting $x = 2$ in $y = 2x^2 - 8x + 4$]
$\therefore\ y = -4$	

$\therefore$ co-ordinates of the T.P. are (2, -4).

$y = 2x^2 - 8x + 4$ is a quadratic function and its graph is a parabola.

The y intercept is (0, 4) and from the diagram (2, -4) is a local minimum.

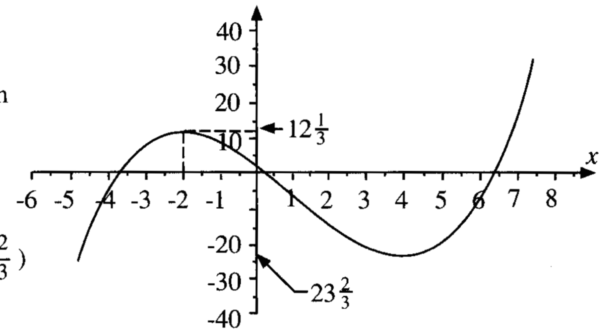


Example C: Find the turning points of $y = \frac{1}{3}x^3 - x^2 - 8x + 3$ and determine their nature. For what set of values is this function decreasing?

Solution:

$y' = x^2 - 2x - 8$	[differentiating y]
$y' = 0$ at any T.P.	
$\therefore\ x^2 - 2x - 8 = 0$	
$\therefore\ (x - 4)(x + 2) = 0$	[factorising]
$x = -2$ or 4	
when $x = -2$, $y = \frac{1}{3}(-2)^3 - (-2)^2 - 8(-2) + 3$	[substituting $x = -2$ in y]
$= -\frac{8}{3} - 4 + 16 + 3$	
$= 12\frac{1}{3}$ $\quad\therefore$ T.P. is $(-2, 12\frac{1}{3})$	
when $x = 4$, $y = \frac{1}{3} \times 4^3 - 4^2 - 8 \times 4 + 3$	[substituting $x = 4$ in y]
$= 21\frac{1}{3} - 16 - 32 + 3$	
$= -23\frac{2}{3}$ $\quad\therefore$ T.P. is $(4, -23\frac{2}{3})$	

Substituting $x = 0$ in $y = \frac{1}{3}x^3 - x^2 - 8x + 3$ shows that the function has a y intercept of (0, 3). A sketch of the graph shows that $(-2, 12\frac{1}{3})$ is a local maximum and $(4, -23\frac{2}{3})$ is a local minimum.



The function is *decreasing* for all values in the interval $-2 < x < 4$.

Use of Calculators

A student with a graphical calculator such as a Casio 6200G can, by drawing the graphs on their calculators, readily see which points are maxima and which are minima. They can also easily tell where the graph is increasing and decreasing.

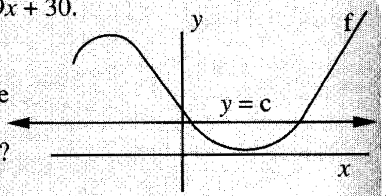
Exercise 21

For each of the following functions, find the co-ordinates of the turning points and determine their nature. State the values for which the functions are increasing and those for which the functions are decreasing.

1. $y = x^2 - 4x + 1$
2. $y = 6 - 8x - x^2$
3. $y = 5 + 3x - 2x^2$
4. $y = 4 - 3x - 2x^2 - \frac{1}{3}x^3$
5. $y = 8 - 6x - \frac{5}{2}x^2 - \frac{1}{3}x^3$
6. $y = 2x^3 + 12x^2 + 18x - 1$
7. $y = x^3 - 12x + 2$
8. $y = 27x - x^3 + 2$

9. f is a real number function such that $f(x) = (x + 1)(x - 1)^2 = x^3 - x^2 - x + 1$
   a. Sketch a graph of f, marking the intercepts on the axes.
   b. Differentiate to find $f'(x)$.
   c. Find the values of x that give turning points of f, stating the nature of these turning points.
   d. For what set of values of x is f a decreasing function?
   e. Find f(0).
   f. Find f′(0).
   g. Find the equation of the tangent to the graph of f, at $x = 0$.
   h. What is the remainder when $f(x)$ is divided by x?

10. A function is defined by $f(x) = x^3 - 3x^2 - 9x + 30$.
    a. Find f(3).
    b. Differentiate to find $f'(x)$.
    c. Determine the values of x that give the turning points of f.
    d. For what values of x is $f(x)$ decreasing?
    e. For what values of c will a straight line $y = c$, parallel to the x axis, cut the graph at 3 points?
    f. Find the equation of the tangent to the graph at $x = 1$.



Problems and Investigations

1. A function $y = Ax^2 + Bx + 2$ has a turning point when $x = \frac{2}{3}$. In addition, when $x = 1$ the function has a value of 9. Find A and B.
2. A function $y = Ax^2 + Bx + 4$ has a turning point when $x = \frac{3}{4}$. The point (2, 8) lies on the graph of this function. Find A and B.
3. $y = Ax^2 + Bx + C$. y' has a value of -2 when $x = 1$ and a value of 2 when $x = 2$. In addition the point (1,-1) lies on the graph of this function. Find A, B and C.

22. APPLICATIONS OF DIFFERENTIATION

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 MEASUREMENT AND CALCULUS

- *to determine and use an expression for the rate of change of a variable and apply it to practical situations, such as maximum, minimum, velocity and acceleration*

Velocity - A Rate of Change

When an object is moving, the distance it travels and the time taken can be used to provide information about its **velocity**. Velocity is the *rate* at which the distance from a reference point is changing with respect to time.

The **average velocity** is the average rate at which the distance from a reference point changes over a period of time and corresponds to the **gradient** of a line segment on a distance time graph.

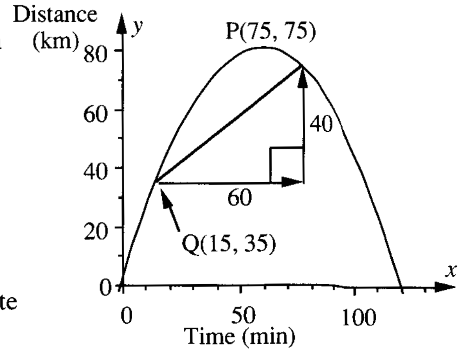
Example A: A car is moving along a road. The distance of the car from a town at various times is shown in the table:

t (minutes)	0	15	30	45	60	75	90	105	120
d (km)	0	35	60	75	80	75	60	35	0

From the 15th to the 75th minute:
the change in distance is 75 – 35 = 40 km and
the change in time is 75 – 15 = 60 minutes

The average velocity of the car from 15th to the 75th minute is the average rate at which the distance changes from the 15th to the 75th minute and is:

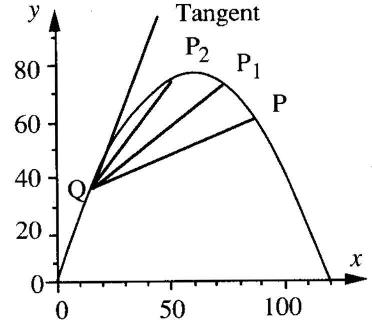
$$\frac{\text{change in distance}}{\text{change in time}} = \frac{75 - 35}{75 - 15} = \frac{40}{60} = \frac{2}{3} \text{ km per minute}$$



The average velocity, $\frac{2}{3}$ km per minute corresponds to the gradient of the line segment joining the points P(75, 75) and Q(15, 35) on the distance-time graph of the car's motion.

The **instantaneous velocity** is the velocity at a particular instant. It is the limiting value of the mean velocity and corresponds to the gradient of the tangent. In a car, the 'speedo' measures the instantaneous *speed* of the car.

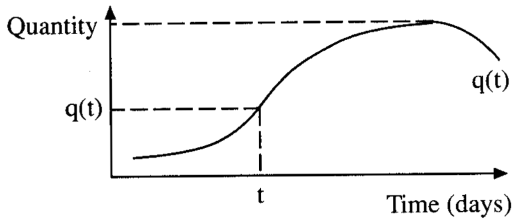
Example B: In Example A, the actual velocity of the car at t = 15 minutes is the instantaneous velocity of the car at t = 15 minutes. It can be found by letting the point P on the diagram shown shift along the curve toward Q through positions P_1 and P_2.



As P gets closer and closer to Q, the mean rate of change of distance between t = 15 and the time corresponding to P will eventually become the gradient of the *tangent* to the graph at (15, 35) which is the **derivative** of the distance graph at (15, 35).

Other Rates of Change

Many other situations involve rates of change. Differentiation is often used to solve such problems in the following way:



The quantity of goods stored in a warehouse is q(t) at time t. The quantity q is measured in kilograms and time t is measured in days. The rate of change at time t can, by a similar argument as that for velocity, be shown to be q'(t), the derivative of q at t.

Example C: The quantity of goodsstored in a warehouse during the year is given by $q(t) = 100 + 30t - 2t^2$ where t is the time in months and q is the quantity of butter measured in tonnes.

a. How much butter is there at the beginning and end of the year?
b. Find the average rate of change of the butter stock during the year.
c. Write an expression for the rate of change of butter stock after t months.
d. At what time was there zero rate of change?

Solution:

a. At the beginning of the year, t = 0 months. The quantity of butter is:
$q(0) = 100 + 30 \times 0 - 2 \times 0^2$ [substituting t = 0 into $q(t) = 100 + 30t - 2t^2$]
$= 100$ tonnes

At the end of the year, t = 12 and the quantity of stock is:
$q(12) = 100 + 30 \times 12 - 2 \times 12^2$ [substituting t = 12]
$= 172$ tonnes

b. The average rate of change during the year is:
$$\frac{\text{quantity at end} - \text{quantity at beginning}}{\text{number of months}} = \frac{172 - 100}{12} = 6 \text{ tonnes per month}$$

c. The rate of change is $q'(t) = 30 - 4t$ tonnes per month [differentiating]

d. When the rate of change is zero, $q'(t) = 0$
$\therefore 30 - 4t = 0$
$\therefore t = 7\frac{1}{2}$ months [solving]

Exercise 22a: Rates of Change

1. The weight of water in a tank over a year varies with time according to the equation $W = 0.007\,t^2 + 0.15\,t + 50$. W is the weight in kilograms of water in the tank and t is the time in days.
   a. What was the initial weight of water?
   b. What was the final weight of water?
   c. What was the mean rate of change in the weight of water during the year?
   d. What was the rate of change after 64 days?
   e. What was the rate of change after 150 days?

2. $S = 0.01t^3 + 0.2t^2 + 5t + 2$ gives the distance in metres of a car from a point at time t in seconds. This formula applies for one minute.
   a. Copy and fill in this table:

t	0	10	20	30	40	50	60
S							

   b. Find the average rate of change of distance during:
      i. the first 10 seconds. ii. the second 10 seconds.
      iii. the first 30 seconds. iv. the full minute.
   c. Find the rate of change:
      i. initially. ii. when t is 10 seconds.
      iii. when t is 30 seconds. iv. when t is 1 minute.

3. The table following shows the amount of coal being mined on various days during a 60 day period. A is the amount of coal mined in a day in tonnes, t is the number of the day being considered.

t	1	21	41	51	60
A	1.4	2.6	3	2.9	2.639

   a. Show that the formula, $A(t) = 1.319 + 0.082t - 0.001t^2$ models the situation described.
   b. How much coal was mined on the 45th day?
   c. Find the mean rate of change per day in the amount of coal mined from the 51st day to the 60th day.
   d. Find the mean rate of change in the amount of coal mined from the 5th day to the 51st day.
   e. Write an expression for the rate of change of the amount of coal mined on day t.
   f. Find the rate of change of the amount of coal being mined on the 11th day.
   g. By drawing a graph or otherwise find which day had the greatest coal production.

4. The length of the side of a cube is (5 + 0.1 t) cm where t is the time measured in minutes. This formula holds true for 10 minutes.
   a. What is the length of the cube's side after 10 minutes?
   b. What is the rate of change of the length of the cube's side after t minutes?
   c. What is the area of a face of the cube after t minutes?
   d. What is the area of a face of the cube after 4 minutes?
   e. What is the total surface area of the cube after 5 minutes?
   f. What is the volume of the cube after t minutes?
   g. What is the volume of the cube after 5 minutes?
   h. What is the rate of change of the volume after t minutes?

5. A rectangle has the length of its base given by 3 + 0.1 t metres and its height given by 1 + 0.2 t metres where t is measured in hours.
   a. What is the initial length of the base of the rectangle?
   b. At what rate is the length of the base increasing?
   c. At what rate is the height increasing?
   d. At what time does the length of the base equal the length of the height?
   e. What is the area of the rectangle at time t?
   f. What is the area of the rectangle after 3 hours?
   g. What is the rate at which the area changes after t hours?
   h. At what time does the rectangle increase in area by 5 m^2 per hour?

6. The length of the base of a rectangle is given by 7 – 0.1 t metres and its height by 2 + 0.2 t metres where t is measured in hours.
   a. What are the rates of change of the length of its base and height?
   b. Explain what the values obtained in a. mean.
   c. What is the time when base and height are the same?

   d. When does the rectangle cease to exist?
   e. What is the area of the rectangle at time t?
   f. What is the rate of change of the area of the rectangle?
   g. What is the initial rate of change of the area of the rectangle?
   h. When is the area increasing by 0.6 m^2 per hour?
   i. When is the area decreasing by 0.3 m^2 per hour?
   j. By sketching a graph or otherwise determine the maximum area.

7. The radius of a circular oil slick is given by 2 + 0.3t cm where t is in minutes. The oil slick is centred in the middle of a circular tank of radius 50 cm.
   a. How much is the radius of the oil slick increasing by per minute?
   b. How much time elapses until the oil slick has expanded as far as possible?
   c. What is the time when the radius of the oil slick is 30 cm?
   d. What is the area of the oil slick at time t?
   e. What is the area of the oil slick after 1 hour?
   f. What is the rate at which the area is increasing after 2 hours?

8. A company allots the sum of $100 000 to market a product. After t months total expenses are 10 000t and total revenue is $250t^2$. Both are in dollars. These relationships apply for 30 months.
   a. Write a formula for M, the total amount of money after t months due to this product.
   b. What is the value of M after 2 months?
   c. What is the rate of change of M after t months?
   d. During what period of time is M decreasing?
   e. Sketch a graph or otherwise to find the minimum value of M.

9. A different company to that in question 8 allots $60 000. Total expenses after t months are 5 000t and total revenue $100t^2$. Both are in dollars. These relationships apply for 3 years.
   a. Write a formula for N, the total amount of money after t months due to this product.
   b. What is the value of N after 3 months?
   c. What is the value of N after 25 months? What is the significance of this?
   d. After what times is N equal to $2 400?
   e. What is the rate at which N is changing after 5 months?
   f. At what time does N have a zero rate of change?

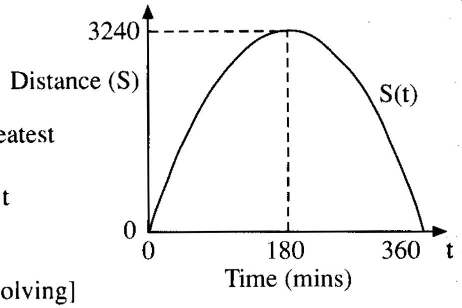
10. A right-angled triangle has its 2 shorter sides of lengths 2 + 0.1t and 3 + 0.2t. Both sides are measured in metres, time is measured in minutes.
   a. How rapidly are the shorter and longer of these sides increasing respectively?
   b. When will the longer side be twice as long as the shorter side?
   c. What is the area of the triangle at time t?
   d. What is the rate of change of area of the triangle at time t?
   e. When is the rate of change of area equal to 2.1 m^2 per minute?
   f. What is the length L of the longest side of the triangle at time t?
   g. Find the rate of change of L^2 with respect to t.

11. A cube has volume $(5-2t)^3$ measured in cm^3 with t measured in minutes.
  a. What is the length of a side of the cube after t minutes?
  b. When does the cube cease to exist?
  c. What is the volume of the cube after 1 minute?
  d. At what time does the cube have a volume of 8 cm^3?
  e. What is the surface area of the cube at time t?
  f. What is the rate of change of the surface area at time t?
  g. What is the rate of change of the volume of the cube at time t?

Maxima and Minima

Turning points can be used to find the maximum or minimum values of a quantity where the quantity varies according to a particular equation.

Example D: The distance S of a boat from a port is given by $S = 36t - 0.1t^2$, where S is in metres and t is in minutes. Find the greatest distance of the ship from the port.



Solution: S is a quadratic function of t. The following sketch of the graph shows an inverted parabola. The maximum corresponds to the greatest distance from the port.
Differentiating S gives $S' = 36 - 0.2\,t$
At the maximum distance, $S' = 0$

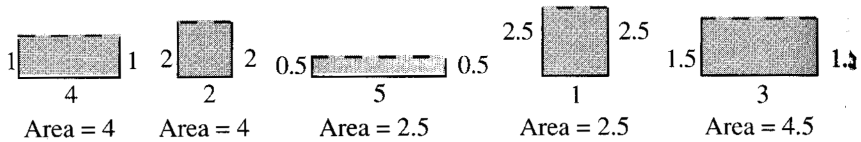
$\therefore 36 - 0.2\,t = 0$

$t = 180$ minutes [solving]

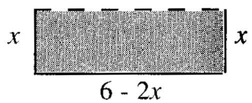
maximum distance $= 36 \times 180 - 0.1 \times 180^2 = 3\,240$m

Example E: Daniel has 6 m of wire. He bends it to give three sides of a rectangular shape so that two of these sides have the same length. Find the length of each side so that the shape has the largest possible area.

Solution: Some possible lengths and the resulting areas are shown:



The sketches suggest a maximum area of 4.5 m^2 when two sides are each 1.5 m and the third 3 m. This can be confirmed using calculus:
Let two sides have a length of x, and the area of the shape be A.



$\therefore$ Length of the third side $= 6 - 2x$

$\therefore$ area, A $= x(6-2x)$ [area is length × breadth]

$= 6x - 2x^2$

$\therefore A' = 6 - 4x$ [differentiating A]

Area, A, is a maximum when $A' = 0$

$\therefore 6 - 4x = 0$

$\therefore x = 1\frac{1}{2}$

$\therefore$ maximum area $A = 1\frac{1}{2}\left(6 - 2 \times 1\frac{1}{2}\right)$ [substituting $x = 1\frac{1}{2}$ into $A = x(6-2x)$]

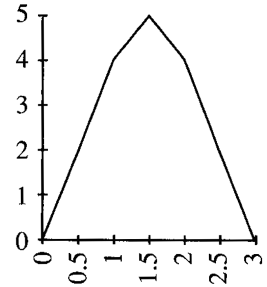
$= 4\frac{1}{2}$ m^2

Use of Spreadsheets

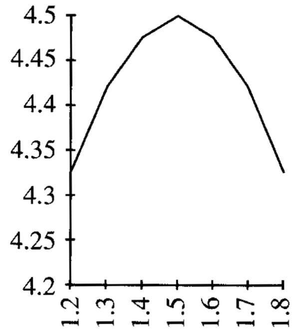
Example E: Done using a spreadsheet:

Area of rectangle problem

x	$6-2x$	Area
0	6	0
0.5	5	2.5
1	4	4
1.5	3	4.5
2	2	4
2.5	1	2.5
3	0	0



x	$6-2x$	Area
1.2	3.6	4.32
1.3	3.4	4.42
1.4	3.2	4.48
1.5	3	4.5
1.6	2.8	4.48
1.7	2.6	4.42
1.8	2.4	4.32



This pair of worksheets and charts show how a maximum or minimum problem can easily be tackled using a spreadsheet. This set of diagrams was created in about four minutes using the Excel spreadsheet. The first worksheet suggests the next set of x values to use in the search for that value of x which gives the maximum area. In this case it is easy to see the answer is 1.5. In other cases though the value may be some decimal whose accuracy can be made as precise as is necessary by doing repeated spreadsheets.

Example F: Find the minimum positive value of $x + y$ if $xy = 9$.

Solution: Let $V = x + y$

$y = \frac{9}{x}$ [making y the subject of $xy = 9$]

$\therefore V = x + \frac{9}{x}$ [substituting into $V = x + y$]

For a minimum value, $V' = 0$

$\therefore 1 - \frac{9}{x^2} = 0$ [differentiating V and putting V = 0]

$\therefore x^2 - 9 = 0$ [rearranging]

$\therefore x = 3$ [solving for a positive value of x]

and $y = 3$ [substituting into $y = \frac{9}{x}$]

$\therefore V = 6$

$\therefore$ the minimum value of $x + y$ is 6.

Exercise 22b: Maxima and Minima

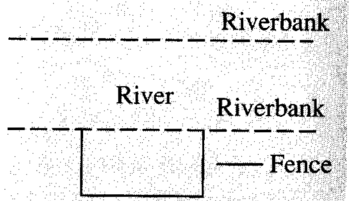
1. The distance D of a boat from a port in metres is given by $D = 48t - 0.08\,t^2$ where t is in minutes. Find the maximum distance of the boat from the port.

2. The volume of water in a tank in litres is given by $V = 360t - 6t^2$ where t is measured in minutes.
   a. Find the maximum volume of water in the tank.
   b. At what time is the maximum volume reached?

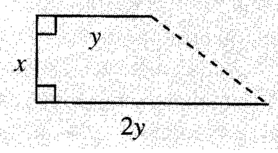
3. The speed of a boat in kilometres per hour is given by $12t^2 - 60t + 90$ over a period of 6 hours (t is in hours). For this six hour period find:
   a. The maximum speed.
   b. The minimum speed.

4. The depth of water at a pier at t hours after sunrise is given by:
   $d = \frac{1}{3}t^3 - 12t^2 + 95t$ where d is measured in centimetres from a mark on the pier.
   a. When is high tide and when is low tide?
   b. At these times what is the depth of water from the mark on the pier?

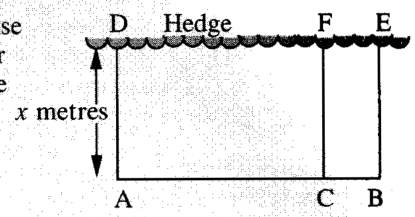
5. What is the maximum possible area bounded by a fence of length 100 m if the fence forms a rectangle with a river bank which is not fenced, as shown in the diagram?



6. a. What is the maximum possible area of a trapezium made by bending a piece of wire 10cm long, as shown in the diagram? (4th side, not wire, shown dotted) [Note the two right angled bends.]

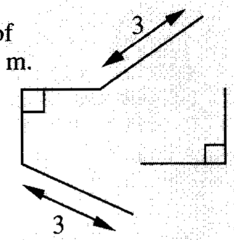


   b. A farmer is about to build two holding paddocks alongside a dense hedge, one for sheep, the other for cattle as shown in the diagram. He has only 240 metres of fencing available, but he wants to enclose the largest possible area.



   i. Suppose the width of the strip of land is x metres. Show that the length AB is $(240 - 3x)$ metres and write down an expression for the total area, A m^2, of the two paddocks.
   ii. Find a value of x for which this area is a maximum, and find the maximum area.

7. A farmer constructs a rectangular enclosure by means of a fence of total length 150 m and two gates of length 3 m. These are situated as shown in the following diagram. Find the maximum area enclosed.

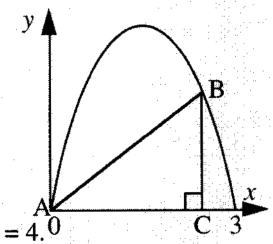


8. What is the maximum value of XY if X + 2Y = 10?

9. What is the maximum value of UV if 3U + 2V = 12?

10. What is the maximum value of X^2Y if X +Y = 6?

11. A right-angled triangle ABC is drawn as shown so A is at (0, 0), B is on the parabola $y = 3x - x^2$, and C is on the x axis. Find the maximum possible value of the area of this triangle.



12. a. Find the minimum positive value of $x + 2y$ if $xy = 4$.
    b. Find values of x and y so that $x + 2y$ exceeds 1 000.

13. Find the minimum positive value of $2x + 3y$ if $xy = 5$.

14. Find the minimum positive value of $x^2 + y^3$ if $xy = 4$.

15. Find the minimum perimeter of a rectangle if its area is 10.

16. A box-shaped object is made by moulding a piece of putty of volume 125 cm^3. It has a square base. Find its minimum surface area.

17. $x = 7 - 3t$, $y = 2 + 4t$, $s = 2xy$
    Find the maximum value of s, and the values of x and y when s is a maximum.

18. Find the maximum and minimum values of $x^2 - 4x - 12$ on the set $-3 \le x \le 8$.
[A function assumes its maximum and minimum values on a closed interval either at turning points or at the ends of the interval.]

19. Find the maximum and minimum values of $x^3 - 12x$ on the interval $-3 \le x \le 6$.

Problems and Investigations

1. Investigate the problem of cutting a piece of string into two pieces. One will be turned into a square, the other into a circle and in so doing, cover:
   a. the least area. b. the greatest area.

2. Repeat question 1 where this time the pieces will be turned into a square and an equilateral triangle.

3. A piece of string is cut into n equal pieces, each of which is formed into an equilateral triangle. Investigate the area of the combined n areas with a view to minimising and maximising total area.

23. BASIC TRIGONOMETRY

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 ESTIMATING AND MEASURING;

- *to model a given situation using trigonometry to find and interpret measures in context, and evaluate their findings.*

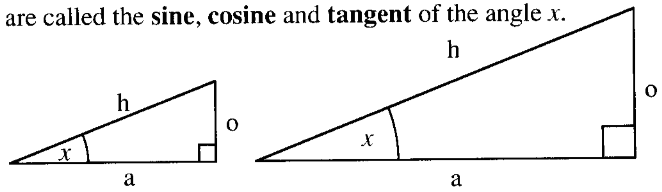
Sine, Cosine and Tangent

Consider the angle x in the right-angled triangle shown. If the size of angle x stays the same while the size of the triangle changes, the values of $\frac{o}{h}$, $\frac{a}{h}$ and $\frac{o}{a}$ remain the same. These values are called the **sine**, **cosine** and **tangent** of the angle x.

$\frac{o}{h}$ = sine x

$\frac{a}{h}$ = cosine x

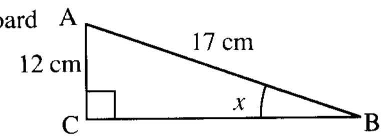
$\frac{o}{a}$ = tangent x



Note: o refers to the side *opposite* angle x.
a refers to the side *adjacent* to angle x.
h refers to the *hypotenuse*, the side opposite to the right angle.
The hypotenuse is always the longest side of a right-angled triangle.

The names are commonly abbreviated to sin x, cos x and tan x.

Example A: Angela cuts a piece of cardboard as shown in the diagram and measures the lengths of two sides. Find the angle $A\hat{B}C$.



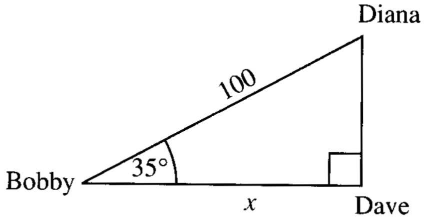
Solution: Let the angle $A\hat{B}C$ be x. Since 12 cm is opposite to $A\hat{B}C$ and 17 cm is the hypotenuse: $\sin x = \frac{12}{17}$ [substituting o = 12, h = 17 in $\sin x = \frac{o}{h}$]
$= 0.70588$
$\therefore x = 44.9°$ [using the $\sin^{-1}$ key on a calculator]

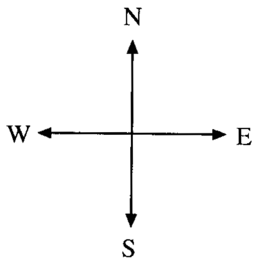
Note: On a calculator, x is found from $\sin x = \frac{12}{17}$, using the keys:

12 → ÷ → 17 → = → inv → sin

Example B: Bobby, Dave and Diana are on a field. Diana is 100m E 35° N of Bobby. She is due North of Dave who is East of Bobby. How far is Dave from Bobby?

Solution: It is always wise to draw a diagram for problems like this:





Let the distance between Bobby and Dave be x.

x is adjacent to 35°.
100 is the hypotenuse.

$\therefore \cos 35° = \frac{x}{100}$ [substituting x = a, 100 = h in cosine = $\frac{a}{h}$]

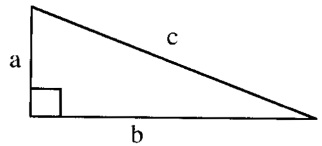
$\therefore x = 100 \cos 35°$ [multiplying by 100]

$\therefore x = 81.9$m (1 d.p.)

Pythagoras' Theorem

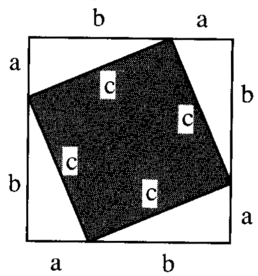
In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares of the other two sides.

In the right-angled triangle shown with sides a, b and c:



$a^2 + b^2 = c^2$

Proof: To prove that $c^2 = a^2 + b^2$, two squares are constructed as shown.



The area of the large square can be found by multiplying the lengths of the sides together:
area = (a + b) × (a + b) = $(a + b)^2$.

The area of the large square can also be found by adding together the areas of the shaded square and the four triangles: area = $c^2 + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab$.

The two areas are the same so:

$c^2 + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab = (a + b)^2$

$\therefore c^2 + 2ab = a^2 + 2ab + b^2$
[simplifying and expanding]

$\therefore c^2 = a^2 + b^2$
[subtracting 2ab from both sides]

Q.E.D.

Example C: The length of the side BC in Example A can be found using Pythagoras' theorem.

$AB^2 = AC^2 + BC^2$

$\therefore 17^2 = 12^2 + y^2$ [letting y be the length of BC]

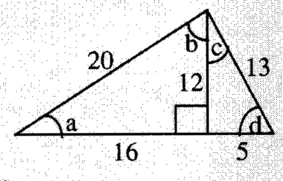
$\therefore 17^2 - 12^2 = y^2$ [subtracting 12^2]

$\therefore 145 = y^2$ [simplifying]

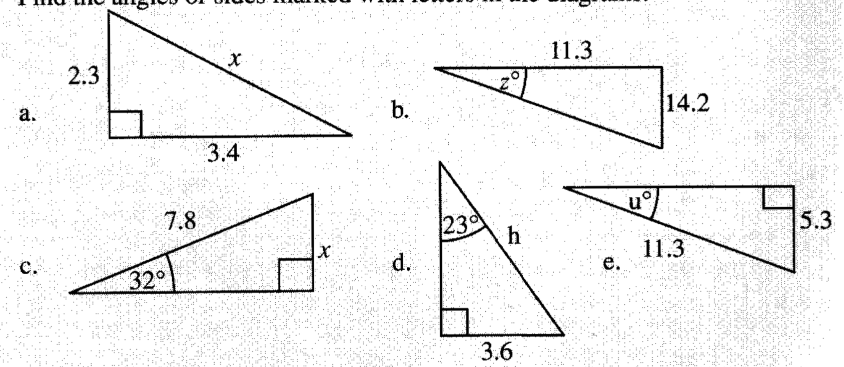
$\therefore y = 12.04$ cm (2 d.p.)

Exercise 23a: Basic Trigonometric Problems

1. Use the diagram to answer the questions:
   a. cos a = b. sin a = c. cos b =
   d. sin b = e. tan b = f. sin c =
   g. tan a = h. sin d = i. tan d =



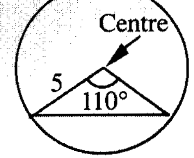
2. Find the angles or sides marked with letters in the diagrams:



3. A ship sails 5 km due west and then 3 km due north. Find the angle which the line joining the ship to its original position makes with the north.

4. To find the distance to a hill C, two observers A and B position themselves so that AC is at right angles to AB. If AB is 15 km and the angle ABC is 49°, find AC.

5. A chord **subtends** an angle of 110° at the centre of a circle of radius 5 m. Find the length of the chord.



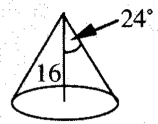
6. The sides of a rectangle are 8 cm and 10 cm long. Find the acute angle between the diagonals.

7. A straight, sloping road runs beside a horizontal wall. A man 2 m high finds that his head is level with the wall at one point of the road and that his feet are level with the wall after he has walked 65 m further up the road. Find the angle of the slope of the road.

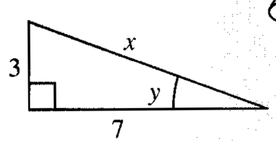
8. An extending ladder slopes at an angle of 55° when its length is 8 m and it is placed against a vertical wall.
   a. How high up the wall does it reach?
   b. How far is its foot from the base of the wall?
   c. If it is now extended to 12 m, what is the new angle of slope, if the foot is in the same position?

9. AB is a diameter of a circle with radius 9 cm and AC is a chord which makes an angle of 24° with the diameter. Find the length of the chord.

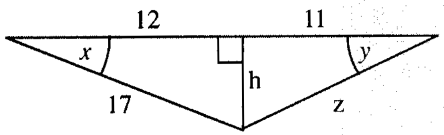
10. a. A cone of height 16 cm has a semi-vertical angle of 24°. Find the length of a slant edge.
    b. Find the diameter of the base.



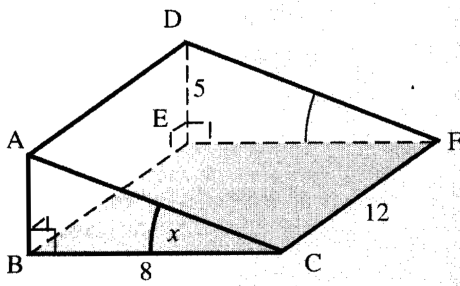
11. Find: a. x
    b. y in the diagram shown.



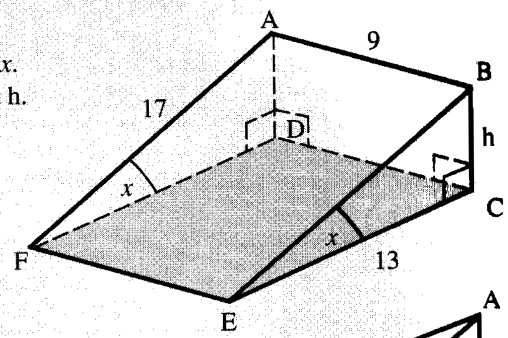
12. In this diagram, find:
    a. x b. h
    c. y d. z



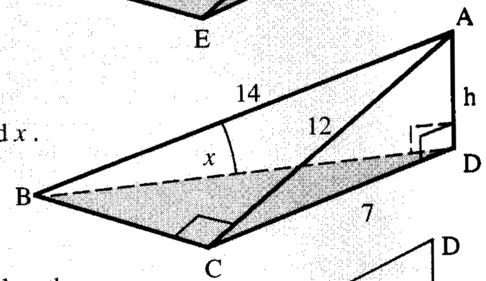
13. In this diagram, find:
    a. the size of $A\hat{C}B$, marked x.
    b. the length of BF.
    c. the length of AC.
    d. the length of AF.
    e. the size of $A\hat{F}B$.



14. In this diagram, find:
    a. the size of $B\hat{E}C$, marked x.
    b. the length of BC, marked h.
    c. the size of $B\hat{F}A$.
    d. the length of AE.
    e. the size of $A\hat{E}D$.
    f. the size of $D\hat{E}C$.

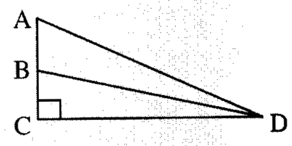


15. In the diagram shown, find:
    a. the length of h.
    b. the size of the angle marked x.
    c. the size of $A\hat{B}C$.
    d. the length of BC.



16. AD is of length 15 m, BC is of length 5 m and BD is 9 m. Find the size $D\hat{A}C$.

17. In this diagram $B\hat{D}C$ is 20°, $A\hat{D}B$ is 30°, BD is 6.8 m. Find the length of AB.



18. A girl stands 15m from a wall and has to look up at an angle of 40° to the horizontal in order to see the top of the wall. How far would she have to walk back from her present position to be able to see the top of the wall while looking up at an angle of 20° to the horizontal?

19. A boy sits at the top of a tree of height 20 m. His two friends are both due north of the tree. The boy has to look down at an angle of 60° to the vertical to see one, and 40° to the vertical to see the other. How far apart are his two friends?

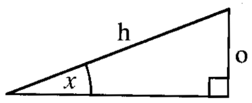
20. A surfcaster looks directly out to sea and sees a fishing boat on the horizon, which is about 16 km away, moving parallel to the coast. A short while later she observes the boat 30° to her right, and 15 minutes after that 40° to her right. How far has the fishing boat travelled in that fifteen minutes?

Other Trigonometric Relationships

For *all angles x*:

$\sin^2 x + \cos^2 x = 1$

Note: $\sin^2 x$ means $\sin x \times \sin x$ or $(\sin x)^2$.



Proof (for any **acute** angle): Consider the right-angled triangle shown:

$\sin^2 x + \cos^2 x = (\sin x)^2 + (\cos x)^2$ [since $\sin^2 x = (\sin x)^2$ and $\cos^2 x = (\cos x)^2$]

$= \frac{o^2}{h^2} + \frac{a^2}{h^2}$ [substituting $\frac{o}{h}$ for $\sin x$ and $\frac{a}{h}$ for $\cos x$]

$= \frac{o^2 + a^2}{h^2}$ [simplifying]

$= \frac{h^2}{h^2}$ [since $o^2 + a^2 = h^2$ by Pythagoras' theorem]

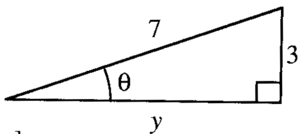
$= 1$ Q.E.D

Two other relationships which are true for *all angles* are:

$\tan x = \frac{\sin x}{\cos x}$ and $\sin x = \cos(90 - x)$

Finding Exact Values of Sine, Cosine and Tangent

Example D: In the triangle shown, $\sin\theta = \frac{3}{7}$. The side y, adjacent to θ, can be found using Pythagoras' theorem:



$y^2 + 3^2 = 7^2$ [by Pythagoras' theorem]

$\therefore y^2 + 9 = 49$

$\therefore y^2 = 40$ [subtracting 9]

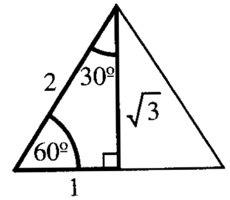
$\therefore y = \sqrt{40}$

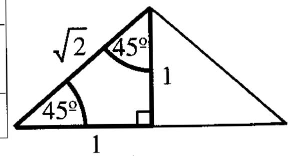
$\cos\theta = \frac{y}{7} = \frac{\sqrt{40}}{7}$ $\tan\theta = \frac{3}{y} = \frac{3}{\sqrt{40}}$

Note: The values, $\frac{\sqrt{40}}{7}$ and $\frac{3}{\sqrt{40}}$ are in **surd** form and are exact.

Applying a similar method to the triangles below gives the exact results shown for the angles 30°, 45° and 60°. The values for 90° are also given.

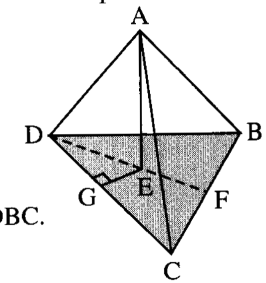
Angle (in degrees)	Sin	Cos	Tan
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	Undefined





Example E: ABCD is a pyramid in which the base is an **equilateral** triangle. The triangle has three sides of length 2 and the height, AE, is also equal to 2. Find in surd form the following:

a. the length of ED. b. the length of AD.
c. the length of EF. d. $\sin A\hat{C}E$
e. the tangent of the angle between planes ABC and DBC. Let G be the point where the line BE meets DC.



Solution:

a. In the ΔDBC, $D\hat{E}C$ is 120° and ED = EC = EB [by symmetry]

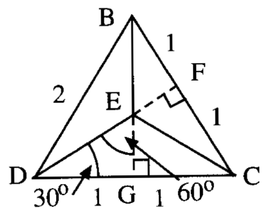
DG = 1 [DG is $\frac{1}{2}$ DC]

DG = ED cos 30º [since $\cos 30º = \frac{DG}{ED}$]

$\therefore 1 = ED \cos 30º$

$\therefore 1 = ED\frac{\sqrt{3}}{2}$ [$\cos 30º = \frac{\sqrt{3}}{2}$]

$\therefore ED = \frac{2}{\sqrt{3}}$ [simplifying]



b. $(AD)^2 = (ED)^2 + (AE)^2$ [using Pythagoras' theorem]

$\therefore AD^2 = \left(\frac{2}{\sqrt{3}}\right)^2 + 2^2$ [since AE = 2 and ED = $\frac{2}{\sqrt{3}}$]

$= \frac{16}{3}$

$\therefore AD = \frac{4}{\sqrt{3}}$ or $\frac{4\sqrt{3}}{3}$ [taking square roots]

c. EF can be found by subtracting DE from DF.

$DC^2 = CF^2 + DF^2$ [applying Pythagoras' theorem to triangle DFC]

$\therefore\ 4 = 1 + DF^2$ [since DC = 2 and CF = 1]

$\therefore\ DF = \sqrt{3}$

$\therefore\ EF = DF - DE = \sqrt{3} - \frac{2}{\sqrt{3}}$ [since $DF = \sqrt{3}$, $DE = \frac{2}{\sqrt{3}}$]

$= \frac{\sqrt{3}\sqrt{3} - 2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$ [changing to a single fraction]

d. $\sin(A\hat{C}E) = \frac{AE}{AC}$ [applying $\frac{o}{h} = \sin x$ to triangle AEC]

$\therefore\ \sin(A\hat{C}E) = 2 \div \frac{4}{\sqrt{3}}$ [since AE = 2, $AC = AD = \frac{4}{\sqrt{3}}$]

$\therefore\ \sin(A\hat{C}E) = \frac{\sqrt{3}}{2}$

e. The tangent of the angle between planes ABC and DBC is:

$\tan(A\hat{F}E) = \frac{AE}{EF}$

$= 2 \div \frac{1}{\sqrt{3}} = 2\sqrt{3}$ [since AE = 2, $EF = \frac{1}{\sqrt{3}}$]

Exercise 23b

Answers to this exercise should be left as surds.

1. In a right-angled triangle one of the angles has a sine of $\frac{1}{3}$. Find the cosine and tangent of this angle.

2. In a right-angled triangle one of the angles has a cosine of $\frac{2}{5}$. Find the sine and tangent of this angle.

3. A right-angled triangle has an angle of size α. Find values for the following, if $\sin\alpha = \frac{1}{4}$.

   a. $\cos\alpha$ b. $\sin(90 - \alpha)$ c. $\tan\alpha$
   d. $\tan(90 - \alpha)$ e. $\sin^2\alpha + \cos^2\alpha$

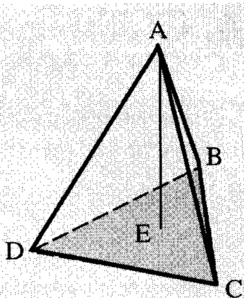
4. A right-angled triangle has hypotenuse of length 5 and an angle α with $\cos\alpha = \frac{1}{4}$. Find the lengths of the two other sides.

5. The shortest side of a right-angled triangle is 3 units long. One of its angles is θ where $\tan\theta = \frac{1}{3}$. Find the lengths of the other two sides.

6. The largest non-hypotenuse side of a right-angled triangle is 5 units long. One of its angles is 60°. Find the lengths of the other sides.

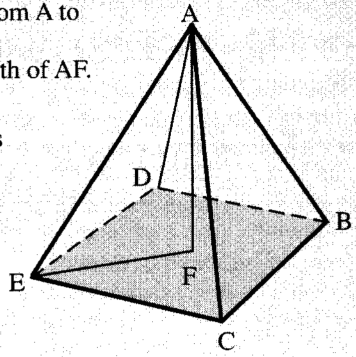
7. ABCD is a pyramid with all sides 2 units long. AE is the perpendicular from A to the plane DBC. Find:

   a. The length of DE. b. The length of AE.
   c. $\sin A\hat{B}E$.
   d. The tangent of the angle between planes ABC and DBC.



8. ABCDE is a square based pyramid with all sides 2 units long. AF is the perpendicular from A to the plane CBDE. Find:

   a. The length of EF. b. The length of AF.
   c. $\sin(A\hat{E}F)$.
   d. The sine of the angle between planes ABC and DBCE.



Problems and Investigations

1. Investigate each of the following statements by substituting values for α to see if they always seem to be true. If they are, prove them.

   a. $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ b. $\sin(90 - \alpha) = \cos\alpha$
   c. $\cos(90 - \alpha) = \sin\alpha$ d. $\sin(\alpha + \beta) = \sin\alpha + \sin\beta$
   e. $\sin(2\alpha) = 2\sin\alpha$ f. $1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}$
   g. $1 + \frac{1}{\tan^2\alpha} = \frac{1}{\sin^2\alpha}$ h. $\tan^2\alpha\cos^2\alpha - \sin^2\alpha = 0$
   i. $\tan^2\alpha\cos^2\alpha + \cos^2\alpha = 1$ j. $\sin\alpha + \cos\alpha = \tan\frac{\alpha}{2}$

2. A sphere of radius R encloses a cube in which the corners are points on the surface of the sphere. Write an expression for the volume of the cube in terms of R.

24. COSINE, SINE AND AREA RULES

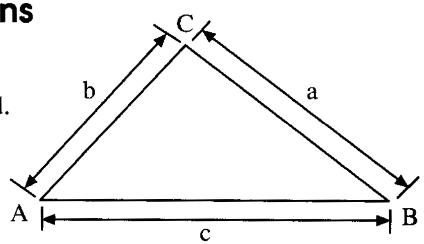
ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

- LEVEL 7 MEASUREMENT;
  - *to model a given situation using trigonometry.*

Introduction and Conventions

In this chapter trigonometry is extended to triangles which are not always right-angled.

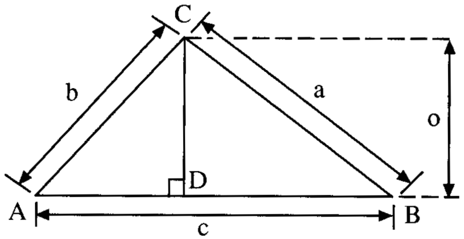


The diagram shows how capital letters are used to show the angles at the corners of the triangle. The length of the side *opposite* each corner is denoted by the same letter in the lower case.

The Cosine Rule

For any triangle the **cosine rule** states:

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Proof (for angle A acute): In triangle ABC, A is acute. The line from C meets AB at right angles at D. The length of CD is o.

In triangle ACD, $\sin A = \frac{o}{b}$ [o is opposite to A, *b* is the hypotenuse]

$\therefore$ CD = b sin A [multiplying by *b*, b = length CD]

Similarly, AD = b cos A

In triangle CDB, BD = AB – AD

$\therefore$ BD = c – b cosA

$CB^2 = CD^2 + BD^2$ [by Pythagoras' theorem]

$\therefore\ a^2 = (b\sin A)^2 + (c - b\cos A)^2$ [substituting]

$\therefore\ a^2 = b^2\sin^2 A + c^2 - 2bc\cos A + b^2\cos^2 A$ [expanding]

$\therefore\ a^2 = b^2\sin^2 A + b^2\cos^2 A + c^2 - 2bc\cos A$ [rearranging]

$\therefore\ a^2 = b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc\cos A$ [factorising]

$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A$ [$\sin^2 A + \cos^2 A = 1$]

Q.E.D.

Note: Although this proof applies only to acute angles, the relationship $a^2 = b^2 + c^2 - 2bc\cos A$ applies to obtuse angles as well.

Making cos A the subject of $a^2 = b^2 + c^2 - 2bc\cos A$ gives the useful result:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

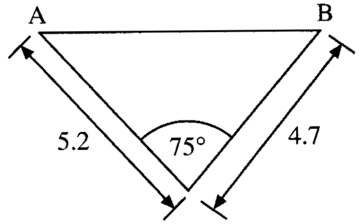
Example A: A fishing vessel sets out to sea. It is 5.2 km from point A on the shore and 4.7 km from another point, B. The angle between the boat and these two points is 75°.

a. How far apart are the two points?

b. Failing to catch any fish the captain sails the boat to a point which is now 5.4 km from point A and 6.3 km from B. What is the angle between A, the boat, and B?

Solution:

a. The diagram shows the points A and B.

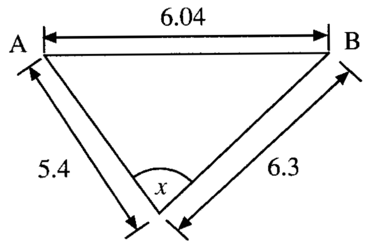


$AB^2 = 5.2^2 + 4.7^2 - 2 \times 5.2 \times 4.7 \cos 75$ [using the cosine rule]

$= 36.48$

$\therefore$ AB = 6.0 km (1 d.p.)

b. The required angle, *x*, is shown below.



$\cos x = \frac{6.3^2 + 5.4^2 - 6.04^2}{2 \times 5.4 \times 6.3}$

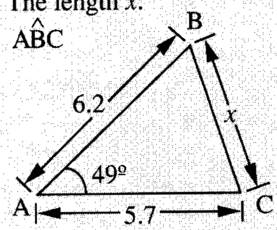
[substituting into $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$]

$= 0.4757$

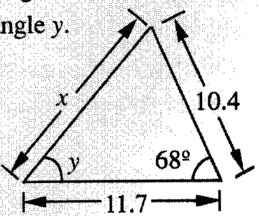
$\therefore x = 61.6°$ (1 dp)

Exercise 24a: Applications

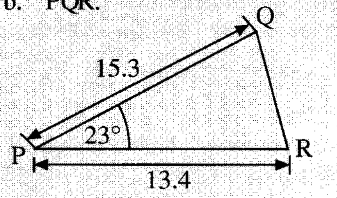
1. In the diagram shown, find:
   a. The length *x*.
   b. $A\hat{B}C$



2. In this diagram find:
   a. The length *x*.
   b. The angle *y*.



3. In this diagram find:
   a. QR.
   b. $P\hat{Q}R$.



4. A bar of length 2m is suspended by two strings of 1.3m and 1.8m long, attached to a nail:
   a. Find the angle between the two pieces of string.
   b. If the bar is now shortened so that the angle between the two strings is 75°, find the new length of the bar.

5. A triangular shaped field has one boundary extending 350 m from a gate in a direction N35°E. The other boundary extends N32°W from the gate for 200m.
   a. Find the length of the third side of the field to the nearest metre.
   b. Find the largest angle of the field.

The Sine and Area Rules

The **sine rule** states:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The **area rule** states that:

$$\text{Area} = \frac{1}{2}ab\sin C$$

Note: Alternatively, the area is equal to $\frac{1}{2}$ bc sin A or $\frac{1}{2}$ ac sin B.

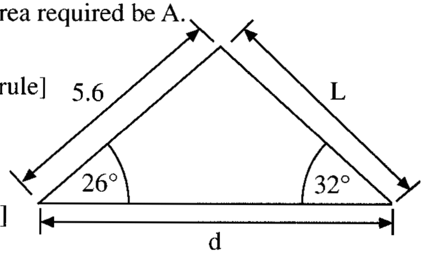
Example B: Peter and Kathy tie two ropes to the top of a pole. They stake the other end of each rope into the ground so they are tight and in the same plane with the pole. Peter's rope is 5.6 m long and makes an angle of 26° with the ground. Kathy's rope makes an angle of 32° with the ground.
a. Find the length of Kathy's rope.
b. Find the distance between the two stakes.
c. Find the area of the triangle formed by the two ropes and the line along the ground which joins the two stakes.

Solution: The following diagram shows Kathy's rope with length L, and the distance between the stakes d. Let the area required be A.

a. $\frac{L}{\sin 26^c} = \frac{5.6}{\sin 32^c}$ [using the sine rule]

$\therefore L = \frac{5.6}{\sin 32^c}\sin 26°$ [multiplying by sin 26°]

$\therefore L = 4.6$ m (1 dp). [simplifying]



b. $\frac{d}{\sin 122^c} = \frac{5.6}{\sin 32^c}$ [using the sine rule and 3rd angle of triangle, 122°]

$\therefore d = \frac{5.6 \times \sin 122^c}{\sin 32^c}$ [multiplying by sin 122°]

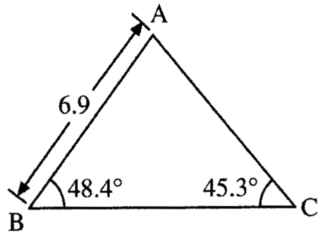
$\therefore d = 9.0$ m (1 dp)

c. $A = \frac{1}{2} \times 5.6 \times 9.0 \times \sin 26$ [using area = $\frac{1}{2}$ ab sin C]

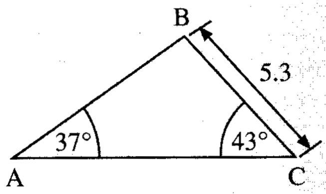
$= 11\ m^2$

Exercise 24b

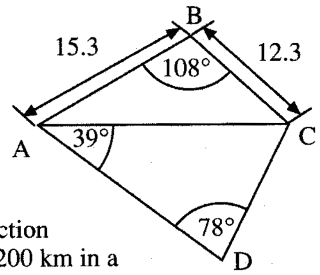
1. In this diagram find:
   a. The length AC.
   b. The length BC.
   c. The area of ΔABC.



2. In this diagram find the length of:
   a. AB
   b. AC



3. In this diagram find:
   a. The length of AC.
   b. The length of CD.
   c. The area of ABCD.

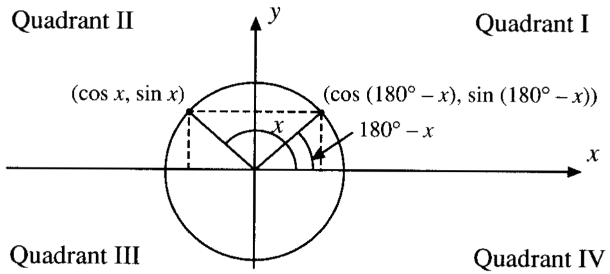


4. A ship sails for 105 km in a direction E32°S. It then sails for another 200 km in a direction N10°E.
   a. How far is it from its original position?
   b. What is its compass bearing from the original position?

5. A father looks down at his son's eyes. The man's eyes are tilted at 38° to the vertical. His son looks at his father's feet. The boy's eyes are tilted at 42° to the vertical. The son's eyes are 1.1 m from his father's eyes. Find the height of the father's eyes from the ground.

6. A triangular wedge has an angle of 43° enclosed between sides of 4.3 cm and 5 cm. Find:
   a. the length of the other side.
   b. the area of the cross-section and hence the volume if the length is 51 cm.

7. A section of land is of triangular shape. One corner of the land has fences of lengths 52 m and 54 m enclosing an angle of 66°.
   a. Find the cost to the nearest dollar of fencing the land at $6.80 a metre.
   b. Find the area of the section.



In the 2nd quadrant the radius (making an angle x with the positive horizontal axis) is the reflection in the y axis of the radius which forms the acute angle $(180° - x)$.

It can be seen from the previous diagram and definitions that:

$\cos x = -\cos(180° - x)$ [since the x co-ordinate of the two points on the circumference corresponding to angles x and $(180° - x)$ are equal and opposite]

$\sin x = \sin(180° - x)$ [since the y co-ordinates of the two points are the same]

$\tan x = \frac{\sin x}{\cos x}$ [by definition]

$= \frac{\sin(180° - x)}{\cos(180° - x)}$ [substituting $\sin x = \sin(180° - x)$ and $\cos x = -\cos(180° - x)$]

$= -\tan(180° - x)$ [simplifying]

The table below shows how the **trigonometric function** of an angle greater than 90° can be expressed as the trigonometric function of an angle in the first quadrant.

Angle	Quadrant	cos x	sin x	tan x
$0° < x \le 90°$	I	$\cos x$	$\sin x$	$\tan x$
$90° < x \le 180°$	II	$-\cos(180° - x)$	$\sin(180° - x)$	$-\tan(180° - x)$
$180° < x \le 270°$	III	$-\cos(x - 180°)$	$-\sin(x - 180°)$	$\tan(x - 180°)$
$270° < x \le 360°$	IV	$\cos(360° - x)$	$-\sin(360° - x)$	$-\tan(360° - x)$

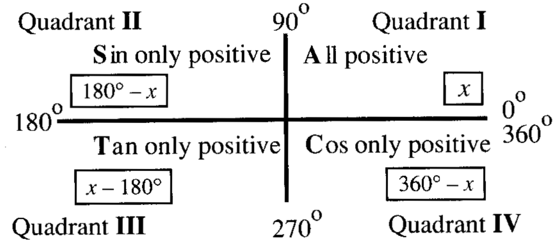
Note: The trigonometrical functions are frequently called the **circular functions** because they are defined by the co-ordinates of the points on a circle.

Example A: $\sin 130° = \sin(180° - 130°)$ [130° in quad II, $\sin x = \sin(180° - x)$]
$= \sin 50°$
$\cos 220° = -\cos(220° - 180°)$ [220° in quad III, $\cos x = -\cos(x - 180°)$]
$= -\cos 40°$

The mnemonic **all science teachers cry** is applied from quadrant I, going anticlockwise to decide which quadrants have positive values for each of the trigonometrical functions:

a refers to quadrant I and stands for *all* are positive.
s refers to quadrant II and stands for *sine only* is positive.
t refers to quadrant III and stands for *tangent only* is positive.
c refers to quadrant IV and stands for *cosine only* is positive.

The following diagram summarises the data from the table above and combines it with the mnemonic.



The sine, cosine and tangent of angles between 90° and 360° can be evaluated using the results summarised on the previous page.

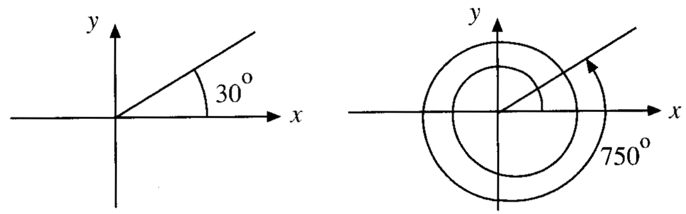
Example B: a. $\sin 135° = \sin(180° - 135°)$ [since 135° in quadrant II]
$= \sin 45°$
$= \frac{1}{\sqrt{2}}$

b. $\tan 330° = -\tan(360° - 330°)$ [since 330° in quadrant IV]
$= -\tan 30°$
$= \frac{-1}{\sqrt{3}}$

Angles Greater than 360°

The trigonometrical functions of angles greater than 360° can be expressed in terms of an acute angle in the following way.

Example C: $750° = 2 \times 360° + 30°$
$\therefore \cos 750° = \cos 30°$
The following diagrams illustrates this:



Exercise 25a

1. Write each of the following as positive or negative sine, cosine or tangent of an acute angle:
[example: sin 312° = -sin 48°]

a.	cos 237°	b.	sin 136°	c.	tan 325°	d.	tan 173°
e.	cos 310°	f.	sin 256°	g.	tan 214°	h.	sin 342°
i.	cos 162°	j.	sin 500°	k.	cos 392°	l.	tan 1060°
m.	cos 1126°	n.	sin 3647°	o.	cos 53°		

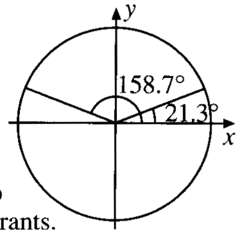
2. Find the values of the following, leaving answers in surd form:

a.	cos 45°	b.	sin 60°	c.	tan 30°	d.	cos 150°
e.	sin 210°	f.	tan 240°	g.	tan 330°	h.	cos 540°
i.	sin 420°	j.	cos 675°	k.	cos 225°	l.	sin 225°
m.	sin 315°	n.	cos 330°	o.	sin 300°		

Trigonometric Equations

A trigonometric equation is given in terms of a circular function. The solution is an angle, usually expressed in degrees.

Example D: Solve $\sin x = 0.363$ for $0° \le x \le 360°$.



Solution: There will be two different angles in the set $0° \le x \le 360°$ for which the sine is 0.363, thus corresponding to $\sin x = 0.363$. These will be in the first and second quadrants because sine corresponds to the y coordinate and is positive only in these two quadrants.

The $\sin^{-1}$ key on a calculator gives 21.3 as follows: (0.363) → (inv) → (sin)

The two angles required are: 21.3° in the 1st quadrant and
$180° - 21.3° = 158.7°$ in the 2nd quadrant.

Example E: Solve the equation $\cos x = -0.4721$ for $0° \le x \le 360°$.

Solution: $\cos^{-1}(0.4721) = 61.8°$ [using the $\cos^{-1}$ key, ignoring the negative.]
Cosine, measuring the x co-ordinate, is negative in the 2nd and 3rd quadrants so the angles required are:
$180° - 61.8° = 118.2°$ in the 2nd quadrant and
$180° + 61.8° = 241.8°$ in the 3rd quadrant.

Example F: Solve the equation $\tan\theta = -1.42$ for $0 \le \theta \le 360°$.

Solution: $\tan^{-1}(1.42) = 54.8°$ [using the $\tan^{-1}$ key, ignoring the negative]
Tangent is negative in the 2nd and 4th quadrants and hence the solutions to the above equation are:

$180° - 54.8° = 125.2°$ in the 2nd quadrant and
$360° - 54.8° = 305.2°$ in the 4th quadrant.

Example G: Solve the equation $\sin(x + 37°) = -0.832$ for $0° \le x \le 360°$

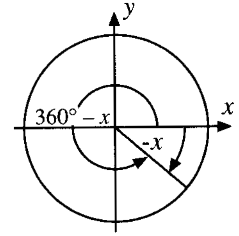
Solution: $\sin^{-1}(0.832) = 56.3°$ [using the $\sin^{-1}$ key, ignoring the negative]
Sine is negative in the 3rd and 4th quadrants:
$\therefore\ x + 37° = 180° + 56.3°$ or $360° - 56.3°$
$= 236.3°$ or $303.7°$ [simplifying]
$\therefore\ x = 199.3°$ or $266.7°$. [subtracting 37°]

Negative Angles

Sometimes a *negative angle* results when solving trigonometric equations.

Example H: Solve the equation, $\cos(x + 49°) = 0.913$ for $0° \le x \le 360°$

Solution: $\cos^{-1}(0.913) = 24.1°$ [using $\cos^{-1}$ key on calculator]
$\therefore\ x + 49° = 24.1°$ [the angle in quadrant I]
or $x + 49° = 360° - 24.1°$ [the angle in quadrant IV]
$\therefore\ x = -24.9°, 286.9°$



Negative angles are the angles obtained by rotating in a *clockwise* direction about the unit circle rather than in an anti-clockwise direction. Thus $-x$ corresponds to the same position on the unit circle as $360° - x$. Hence in Example H, $-24.9°$ is equivalent to $360° - 24.9° = 325.1°$ and so the solutions to Example H should be written $x = 286.9°$ or $325.1°$.

Exercise 25b

Solve these equations where $0° \le x \le 360°$:

1.	$\sin x = 0.3172$	2.	$\cos x = 0.4314$	3.	$\tan x = -1.21$
4.	$\cos x = -0.4621$	5.	$\sin x = -0.8623$	6.	$\tan x = 0.6821$
7.	$\sin(x + 34°) = 0.8634$	8.	$\cos(x - 53°) = -0.7144$		
9.	$2\sin(x - 37°) = 0.8613$	10.	$\frac{4\cos(x + 43°)}{3} = -1.3$		

Equations where the Unknown has a Coefficient ≠ 1

When the unknown has a coefficient other than 1, care must be taken to ensure that *all solutions* are found. There are usually two solutions for each rotation of 360° about the positive horizontal axis. Thus, if $0° \le x \le 360°$, $3x$ takes values in the set $0° \le 3x \le 1\,080°$. This is up to three rotations from the positive horizontal axis, so there will be six solutions to the equation.

Example I: Solve $\tan 3x = 0.5913$ for $0° \le x \le 360°$.

Solution: $\tan^{-1}(0.5913) = 30.6°$ [using a calculator]
Since $0° \le x \le 360°$, $3x$ lies in the set $0° \le x \le 1\,080°$. $[3 \times 360° = 1\,080°]$
There are two solutions for $3x$ in the set 0° to 360°:

$3x = 30.6°$ and $3x = 180° + 30.6° = 210.6°$ [tan positive in quad I & III]

$\therefore\ x = 10.2°$ and $70.2°$ [dividing by 3]

Two other solutions occur when $3x$ lies in the set 360° to 720°. These are obtained by solving:

$3x = 360° + 30.6° = 390.6°$ $\therefore\ x = 130.2°$ [dividing by 3]
$3x = 360° + 210.6° = 570.6°$ $\therefore\ x = 190.2°$ [dividing by 3]

The last two solutions occur when $3x$ lies in the set 720° to 1 080°. These are:

$3x = 720° + 30.6° = 750.6°$ $\therefore\ x = 250.2°$ [dividing by 3]
$3x = 720° + 210.6° = 930.6°$ $\therefore\ x = 310.2°$ [dividing by 3]

$\therefore$ Solutions are $x = 10.2°, 70.2°, 130.2°, 190.2°, 250.2°, 310.2°$

Example J: Solve $\sin 2x = \frac{-1}{\sqrt{2}}$, $0° \le x \le 360°$

Solution: $\sin^{-1}(\frac{1}{\sqrt{2}}) = 45°$ [exact trig. value or by using a calculator]

Since $2x$ lies in the set 0° to 720° and sin is negative in quadrants III and IV:

$2x = 180° + 45°, 360° - 45°, 360° + 180° + 45°, 360° + 360° - 45°$
$\therefore\ x = 112.5°, 157.5°, 292.5°, 337.5°$ [dividing by 2]

Exercise 25c

1. Solve the following equations for $0° \le x \le 360°$:
   a. $\sin 3x = 0.5613$ b. $\cos 2x = 0.2132$
   c. $\tan 3x = 1.3342$ d. $\tan 2x = -0.6345$
   e. $\cos 4x = -0.5632$ f. $\sin 2x = -0.4763$
   g. $\tan 2x = 0.7634$ h. $\sin 4x = 0.8632$
   i. $\sin\frac{x}{2} = 0.1342$ j. $\cos\frac{x}{3} = 0.4152$

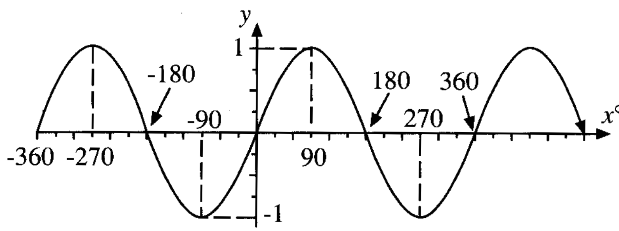
2. Solve the following equations:
   a. $\cos x = 0.3164$, $0° \le x \le 360°$ b. $\sin \theta = 0.4839$, $0° \le \theta \le 360°$
   c. $\sin y = -0.8134$, $0° \le y \le 360°$ d. $\tan x = -2.137$, $0° \le x \le 360°$
   e. $\cos x = 0.135$, $0° \le x \le 360°$ f. $\tan x = 2.3$, $0° \le x \le 540°$
   g. $\cos 3x = 0.3625$, $0° \le x \le 180°$ h. $\cos(x + 30°) = 0.83$, $0° \le x \le 180°$
   i. $3\tan(x + 35°) = 5$, $0° \le x \le 180°$ j. $3\tan 2x = 1$, $0° \le x \le 270°$
   k. $5\cos(x - 70°) = -0.853$, $0° \le x \le 360°$
   l. $5\sin 3x = 2\sin 3x + 0.48$, $0° \le x \le 180°$

3. Solve the following equations for x where $0° \le x \le 360°$:
   a. $\sin x = \frac{-1}{\sqrt{2}}$ b. $\sin x = \frac{\sqrt{3}}{2}$ c. $\tan x = \frac{-1}{\sqrt{3}}$
   d. $\cos x = -1$ e. $\cos x = \frac{1}{\sqrt{2}}$ f. $\tan x = \sqrt{3}$
   g. $\sin x = \frac{-1}{2}$ h. $\sin 3x = 1$ i. $\sin 2x = \frac{1}{2}$
   j. $\cos 2x = \frac{\sqrt{3}}{2}$ k. $\sin(x + 60) = \frac{\sqrt{3}}{2}$ l. $\cos(x - 30) = \frac{-1}{2}$
   m. $\cos(x + 45) = \frac{-1}{\sqrt{2}}$ n. $\sqrt{3}\tan 2x = 3$ o. $2\sin^2 x = 1$

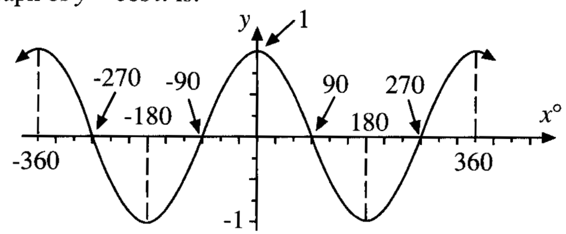
Graphs of the Trigonometrical Functions

The graph of $y = \sin x$ can be plotted from the points in the table.

$x°$	-180	-90	0	90	180	270	360	450	540	630	720
$\sin x°$	0	-1	0	1	0	-1	0	1	0	-1	0

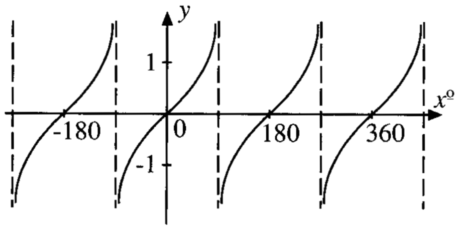


Similarly the graph of $y = \cos x$ is:



The graph of $y = \tan x$ is quite different from the sine and cosine graphs, as seen by plotting the points from the following table.

$x°$	-90	-45	0	45	90	135	180	225	270	315
$\tan x°$	E	-1	0	1	E	-1	0	1	E	-1



Note:

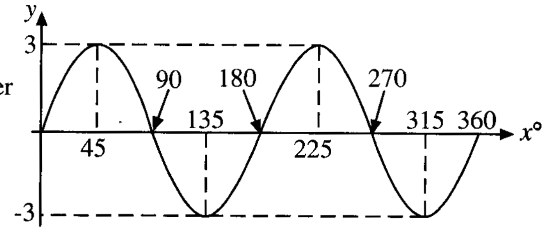
a. The graphs of sine and cosine repeat themselves every 360°. The sine and cosine functions are said to be *periodic* with a **period** of 360°.
b. The **domain** of both sine and cosine is R, the set of all real numbers.
c. The **range** of both sine and cosine is $\{y: -1 \le y \le 1\}$.
d. The period of the tangent function is 180°.
e. The domain of the tangent function is the set of all real numbers except odd multiples of 90° i.e. . . . -270°, -90°, 90°, 270°, 450°, . . .
f. The range of the tangent function is all real numbers.

More Complex Trigonometric Functions

The sketching of graphs of more complex trigonometric functions follows the same patterns as noted in chapter 13.

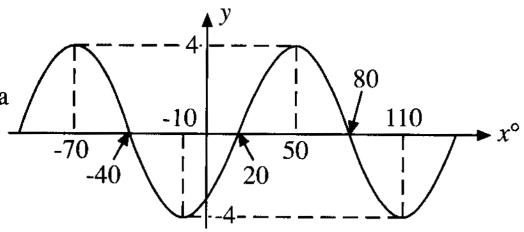
Example K: $y = 3\sin 2x$ is sketched by noting that compared to $y = \sin x$:
The 3 **enlarges** all vertical values by a factor of 3.
The 2 **compresses** the graph by *halving the period* to 180°.

Note: In the case of sine and cosine graphs, the magnitude of the number multiplying the sine or cosine is called the **amplitude**. In example K, the amplitude is 3.



Example L: To sketch $y = -4\cos(3x + 30°)$ consider the effect of -4, 3 and 30° on the graph of $y = \cos x$. The graph is sketched using the information:

a. The '-' in -4 *inverts* the cosine graph and the 4 *stretches* the y values by a factor of 4.
b. The 3 compresses the period to $\frac{1}{3}$ of the normal period ie: to 120°.



c. The function will take the value of -4 when $3x + 30 = 0$, i.e. when $x = -10°$.
d. The function will take the value 0 when $3x + 30 = 90$, i.e. when $x = 20°$.

The relationship of more complicated trigonometrical functions to the basic sine or cosine graphs is summarised below:

In the graphs of $y = \mathbf{A}\sin(\mathbf{B}x + \mathbf{C}) + \mathbf{D}$ or $y = \mathbf{A}\cos(\mathbf{B}x + \mathbf{C}) + \mathbf{D}$

A is the *amplitude* and enlarges the vertical values. (If A < 0, then the graph is inverted)
B changes the *compression* of the graphs, i.e., changes the period to $\frac{360°}{\mathrm{B}}$.
C shifts the y axis to the *right* by $\frac{\mathrm{C}}{\mathrm{B}}$ units, i.e., *the graph moves left* by $\frac{\mathrm{C}}{\mathrm{B}}$ units.
D shifts the x axis *down* by D units, i.e, *the graph moves up* by D units.

Example M: Solve $\sin(4x + 240°) = 0.5$ $0° \le x \le 180°$

Solution: The period is $\frac{360°}{4} = 90°$ [comparing $\sin 4x$ to $\sin x$]
The two values of $4x + 240°$ in the set $0° \le x \le 360°$ which give a sine of 0.5 are:
$4x + 240° = 30°$ or $4x + 240° = 150°$
$\therefore x = -52.5°$ or $x = -22.5°$ [solving the equations]

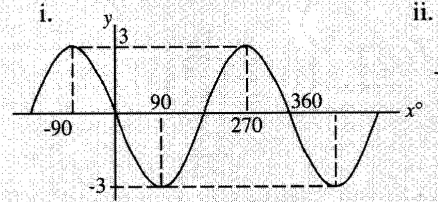
The other solutions are found by adding the period and multiples of the period to -52.5 and -22.5 which gives:
-52.5°, 37.5°, 127.5°, 217.5°, ... and -22.5°, 67.5°, 157.5°, 247.5°, ...
$\therefore$ the solutions in the set $0° \le x \le 180°$ are 37.5°, 67.5°, 127.5° and 157.5°.

Exercise 25d

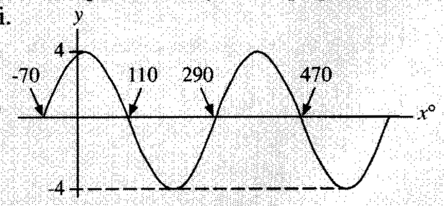
1. Sketch the following graphs:
   a. $y = 2\sin x$ b. $y = -\cos x$ c. $y = \sin x + 1$
   d. $y = \sin(x + 30°)$ e. $y = \sin(x - 30°)$ f. $y = \sin 2x$
   g. $y = 3\sin 2x$ h. $y = 2\sin(x - 60°)$ i. $y = 3\cos(4x - 120°)$
   j. $y = \tan(x - 30°)$

2. Write down the equations of the functions represented by these graphs:

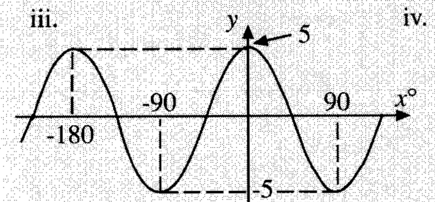
i.



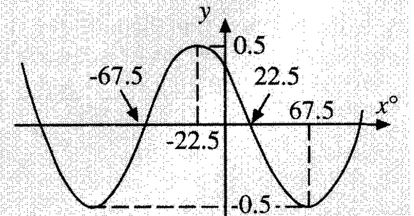
ii.



iii.

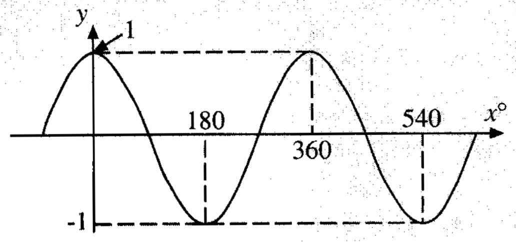


iv.



3. The graph opposite shows which function?

   a. $y = \sin x$
   b. $y = -\sin x$
   c. $y = -\sin(x + 90°)$
   d. $y = \sin(x - 90°)$
   e. $y = \sin(90° - x)$

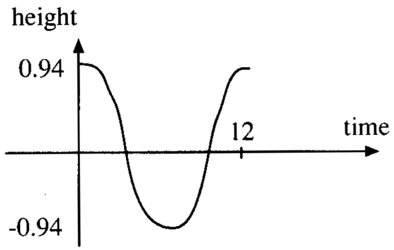


Applications of the Trigonometry Functions:

The trigonometrical functions provide a powerful tool for describing any phenomenon which is periodic.

Example N: The tide reaches high tide every 12 hours in a harbour. The difference between the high and low water marks is 1.88 metres. Write an equation which describes the height of the water as a function of time passed since the last high tide was measured. The height of the water is measured from the position of the horizontal line midway between high and low tides.

Solution: Sketching the situation reveals the following:



The function can be described by a cosine function of amplitude 0.94 $[\frac{1}{2} \times 1.88]$.

To calculate the period in terms of degrees:

12 hours = 360°
∴ 1 hour = 30°
∴ t hours = 30t°
This can now be substituted into the cosine function
∴ the height h(t) of the tide when time is t hours is:

$$h(t) = 0.94 \cos (30t°)$$

Exercise 25e

1. a. Find a function which describes the height of the water since the last occasion the tide was 'midway' between low and high tide for the harbour in example N.
   b. Find a function which describes the height of the water in this harbour since the last low tide.

2. Find a function which describes the depth of the water in a harbour. The depth is 4m at low tide and 6.8m at high tide. Begin at high tide.

3. A wave pattern has a distance of 5m between crests and a vertical distance of 10m between the highest and lowest point on the pattern.
   Find an equation which describes the height of the wave as a function of the distance from the point midway between the lowest and highest points.

4. A wheel is turning about a fixed centre. The wheel has a radius of 6m. It turns through 540° a minute. An ant stuck on the circumference is intially 6m horizontally to the right of the centre of the wheel.
   a. Write an equation which gives the height of the ant above the centre as a function of time.
   b. Find when the ant is 3m above the centre.

5. The depth of water in a harbour at low tide is 5m and at high tide is 7.4m. The bottom of a bridge is 4m above the high tide mark. A boat, the top of whose aerial is 5m above the surface of the water, comes into the harbour at low tide. What is the longest time it can stay before it is trapped in the harbour?

Problems and Investigations

Investigate each of the following sequences to find a function <f(n)> which describes each of them:
1. 0, 2, 0, -2, 0, . . .
2. 3, 8, 3, -2, 3, 8, . . .
3. 1, 4, 3, 2, 5, . . .
4. 3, 12, 3, 12, . . .
5. 0, 4, 0, -8, 0, 12, . . .

26. RADIAN MEASURE

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

- LEVEL 7 MEASUREMENT
  - *to model a given situation, using trigonometry (including radian measure)*
- LEVEL 7 ALGEBRA
  - *to choose suitable strategies (including trigonometric) for finding solutions to equations and interpret the results*

Introduction

The formulae for the circumference and area of a circle are $C = 2\pi R$ and $A = \pi R^2$ respectively, where R is the radius. These formulae involve π (pronounced 'pi'), an **irrational** number which has approximate values of $\frac{22}{7}$,3, 3.1, 3.14, 3.142,..., depending on the degree of accuracy required.

The quantity π provides for an alternative way of measuring the sizes of angles where angles are measured in **radians** rather than degrees.

π radians = 180°

Thus 3 radians are approximately 180°, or, more accurately, 3.142 radians are approximately 180°.

Conversion between Radians and Degrees

An angle measured in degrees can be changed to radians and vice versa using the relationship π radians = 180°.

Example A: Change: a. 48° to radians. b. $\frac{2\pi}{15}$ radians to degrees.

Solution: a. 180° = π radians

$\therefore\ 1° = \frac{\pi}{180}$ [dividing by 180]

$\therefore\ 48° = \frac{48 \times \pi}{180}$ [multiplying by 48]

$= \frac{48 \times 3.142}{180}$ [taking π = 3.142 and multiplying]

= 0.838 radians (3 d.p.)

b. π radians = 180°

$\therefore\ \frac{2\pi}{15}$ radians $= \frac{2 \times 180°}{15}$ [dividing by 15 and multiplying by 180]

$= 24°$

Note: Books of mathematical tables contain a section allowing conversion between radians and degrees.

Example B: A wheel of radius 1.5 m does three revolutions in 1 minute.
a. How many radians does the wheel turn through in 45 seconds?
b. How fast does a point on the circumference travel in metres per second?

Solution:
a. The wheel turns through 3 revolutions in 60 seconds.

$\therefore$ In 45 seconds, the wheel turns $\frac{45}{60} \times 3$ revolutions. [using proportion]

$= \frac{45 \times 3 \times 2 \times \pi}{60}$ [since 1 revolution is 2π radians]

$= \frac{45 \times 3 \times 2 \times 3.142}{60}$ [substituting 3.142 for π]

= 14.14 radians (2 d.p.)

b. In 1 minute a point on the circumference travels a distance equal to three times the circumference, which is $3 \times 2\pi R = 6\pi R$ metres.

$\therefore$ Its speed is $\frac{6\pi R}{60}$ metres per second. [speed = $\frac{\text{distance}}{\text{time}}$]

$= \frac{6 \times 3.14 \times 1.5}{60}$ [taking π = 3.14 and R = 1.5 m]

$= 0.5\ ms^{-1}$ (1 dp)

Exercise 26a: Angle Conversion

1. Change the following angles given in degrees into radians (take π = 3.14):
   a. 18° b. 38° c. 156° d. 247° e. 418°
2. Change the following angles, given in radians, into degrees:
   a. 2.6 b. 3.4 c. 1.18 d. 0.57 e. 0.03
3. The following angles are given in radians as multiples or fractions of π. Change them into degrees.
   a. $\frac{2\pi}{3}$ b. $\frac{2\pi}{4}$ c. $\frac{5\pi}{9}$ d. $\frac{3\pi}{8}$ e. $\frac{4\pi}{5}$
   f. $\frac{7\pi}{12}$ g. $\frac{5\pi}{12}$ h. $\frac{\pi}{8}$ i. 3π j. $\frac{5\pi}{6}$
4. Change these angles, given in degrees, into fractions or multiples of π:
   a. 120° b. 135° c. 315° d. 405° e. 22.5°
   f. 72° g. 108° h. 84° i. 132° j. 27°
5. A fly walking round a circle does 5.5 circuits. How many radians will the fly have turned through?

6. The fly in 5. above continues walking and goes through another 37.7 radians. How many more circuits will the fly have made of the circle?

7. A wheel is rotating with a speed of 15 radians per second. How many revolutions is this per second? How many revolutions is this per minute?

8. Another wheel is rotating at 12 revolutions per second. How many radians does the wheel rotate through in a second?

9. A wheel turns through 72 000° a minute. How many radians does it turn through in a second?

10. A spoke rotated through 52 778.75 radians in an hour. What was the number of revolutions it did per second?

11. Mary runs around a circular shaped track.
   a. After running $3\frac{1}{3}$ laps, how many radians will she have turned through?
   b. If Mary has run through 64 radians, how many laps has she done?

12. A string is to be wound around a circular rod of radius 8 mm. Assuming that all the string lies in one layer on the rod, how many turns are necessary in order to wind in 20 m of string?

13. A pulley, 18 cm in diameter, drives another pulley of 12 cm in diameter. The former revolves at 240 revolutions per minute.
   a. What is the angular speed of the 12 cm pulley in revolutions per minute?
   b. What is the angular speed of the 18 cm pulley in radians per second?

14. A pulley, 10.5 cm in diameter, is running at 1 800 revolutions per minute.
   a. What is the angular speed in radians per second of the pulley?
   b. What is the speed of a point on the rim in metres per second?

15. A cog wheel with 36 teeth is engaged to one with 30 teeth. The first rotates at 90 revolutions per minute.
   a. What is the rotational speed of the other wheel in revolutions per minute?
   b. If the radius of the first cog wheel is 10 cm, what is the radius of the second wheel?

Trigonometric Problems

Example C: Find the value of $\sin\frac{4\pi}{3}$ leaving the answer in surd form.

Solution:

$\frac{4\pi}{3}$ radians $= \frac{4}{3} \times 180°$ [changing $\frac{4\pi}{3}$ radians to degrees]

$= 240°$

$\therefore \sin\frac{4\pi}{3} = \sin 240°$

$= -\sin 60°$ [since 240° is in quadrant III]

$= -\frac{\sqrt{3}}{2}$

Example D: Solve the equation $\cos(x + \frac{\pi}{6}) = \frac{1}{2}$, $0 \le x \le 2\pi$.

Solution:

$\cos^{-1}(\frac{1}{2}) = 60°$ [using a calculator]

$60° = \frac{60 \times \pi}{180}$ radians [changing to radians]

$= \frac{\pi}{3}$ radians

Cosine is positive in the I and IV quadrants, so the two solutions are:

$\therefore x + \frac{\pi}{6} = \frac{\pi}{3}$ or $x + \frac{\pi}{6} = 2\pi - \frac{\pi}{3}$ [2π radians = 360°]

$\therefore x = \frac{\pi}{6}$ or $\frac{3\pi}{2}$ [rearranging and solving]

When are Radians Used?

In most practical situations, such as building, navigation and engineering, angles are normally measured in degrees.

However, angles are measured in radians in most advanced areas of pure mathematics and science for a number of reasons.

One reason is that if x is measured in radians: the ratio $\frac{\sin x}{x}$ gets very close to 1, as x gets very small.

This result is very important and is used to establish many other important results in more advanced mathematics.

Exercise 26b

1. Evaluate each of the following, giving the answer exactly:
   a. $\sin\frac{\pi}{2}$ b. $\cos\frac{3\pi}{4}$ c. $\sin\frac{3\pi}{4}$ d. $\tan\frac{3\pi}{4}$
   e. $\tan(\frac{-\pi}{3})$ f. $\sin(\frac{-\pi}{6})$ g. $\cos(\frac{-2\pi}{3})$ h. $\cos 2\pi$
   i. $\sin\frac{4\pi}{3}$

2. Which of the following is a solution to $\sin x = 0.5$?
   a. $\frac{\pi}{2}$ b. $\frac{\pi}{3}$ c. $\frac{\pi}{4}$ d. $\frac{\pi}{6}$ e. 0

3. Which of the following is a solution to $\cos x = \frac{-\sqrt{3}}{2}$?
   a. $\frac{\pi}{3}$ b. $\frac{2\pi}{3}$ c. $\frac{5\pi}{6}$ d. $\frac{\pi}{6}$ e. $\frac{4\pi}{3}$

4. Which of the following sets contains the solutions to $\sin x = \frac{1}{\sqrt{2}}$, $0 \le x \le 2\pi$?
   a. $\left\{\frac{\pi}{4}, \frac{7\pi}{4}\right\}$ b. $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ c. $\left\{\frac{3\pi}{4}, \frac{5\pi}{4}\right\}$ d. $\left\{\frac{5\pi}{4}, \frac{7\pi}{4}\right\}$ e. $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$

5. Which of the following sets contains the solutions to $\sin x = -\frac{1}{\sqrt{2}}, 0 \le x \le 2\pi$?

a. $\left\{\frac{\pi}{4}, \frac{7\pi}{4}\right\}$ b. $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ c. $\left\{\frac{3\pi}{4}, \frac{5\pi}{4}\right\}$ d. $\left\{\frac{5\pi}{4}, \frac{7\pi}{4}\right\}$ e. $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$

6. Which of the following sets contains the solutions to $\tan x = 1, 0 \le x \le 2\pi$?

a. $\left\{\frac{\pi}{4}, \frac{7\pi}{4}\right\}$ b. $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$ c. $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ d. $\left\{\frac{5\pi}{4}, \frac{7\pi}{4}\right\}$ e. $\left\{\frac{3\pi}{4}, \frac{5\pi}{4}\right\}$

7. Find the solutions for the following trigonometrical equations. (Give answers in radians as multiples of π):

a. $\sin x = \frac{-1}{2}, 0 \le x \le 2\pi$ b. $\cos x = \frac{1}{2}, 0 \le x \le 2\pi$

c. $\sin x = \frac{1}{\sqrt{2}}, 0 \le x \le 2\pi$ d. $\tan x = \sqrt{3}, -\pi \le x \le \pi$

e. $\tan x = -1, 0 \le x \le 2\pi$ f. $\sin x = \frac{-\sqrt{3}}{2}, -\pi \le x \le \pi$

g. $\cos x = \frac{1}{\sqrt{2}}, -\pi \le x \le \pi$ h. $\sin (x + \frac{\pi}{4}) = \frac{\sqrt{3}}{2}, 0 \le x \le 2\pi$

i. $\cos (x - \frac{\pi}{3}) = \frac{1}{2}, -\pi \le x \le \pi$ j. $\sin 2x = \frac{\sqrt{3}}{2}, 0 \le x \le 2\pi$

k. $\cos 3x = \frac{-1}{\sqrt{2}}, 0 \le x \le 2\pi$ l. $\sin 3x = 1, \pi \le x \le 2\pi$

m. $\tan 2x = 1, 0 \le x \le 2\pi$ n. $\sin \frac{x}{2} = \frac{1}{2}, 0 \le x \le 2\pi$

o. $\cos \frac{x}{3} = \frac{-1}{\sqrt{2}}, 0 \le x \le \pi$

8. Find solution sets in angles for the following trigonometric equations. Express answers in radians.

a. $\sin x = 0.4, 0 \le x \le 2\pi$ b. $\cos x = 0.3, 0 \le x \le 2\pi$

c. $\tan x = 0.8, 0 \le x \le 2\pi$ d. $\tan x = \frac{-2}{3}, 0 \le x \le 2\pi$

e. $\tan x = 0.6, 0 \le x \le 2\pi$ f. $\tan x = \frac{-1}{2}, 0 \le x \le 2\pi$

g. $3 \sin x = 1, 0 \le x \le 2\pi$ h. $4 \cos x = -3, 0 \le x \le 2\pi$

i. $8 \sin x = 7, 0 \le x \le 2\pi$ j. $-2 \tan x = 3, 0 \le x \le 2\pi$

k. $7 \tan x = -8, 0 \le x \le 2\pi$ l. $-\cos x = 0.1513, 0 \le x \le 2\pi$

Exercise 26c: Further Trigonometric Problems

1. Sketch graphs of the following functions:

a. $y = \cos x, 0 \le x \le 2\pi$ b. $y = 2 \sin x, 0 \le x \le 2\pi$

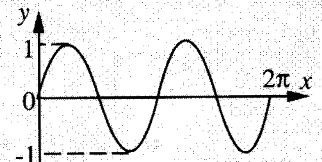
c. $y = \tan x, -\pi \le x \le 2\pi$ d. $y = \sin (x - \frac{\pi}{4}), -\pi \le x \le 2\pi$

e. $y = \sin (x + \frac{\pi}{3}), 0 \le x \le 2\pi$ f. $y = 3\cos (x - \frac{\pi}{2}), 0 \le x \le 3\pi$

g. $y = -2\sin (x + \frac{\pi}{3}), -\pi \le x \le \pi$ h. $y = 2\sin 3x, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

i. $y = 3\cos 2x, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$ j. $y = 4\cos (2x - \frac{\pi}{2}), \frac{-\pi}{2} \le x \le \pi$

2. a. Which function is represented by the graph shown?
   i. $y = 2 \sin x$
   ii. $y = \sin 2x$
   iii. $y = \sin x$
   iv. $y = 2 \cos x$
   v. $y = \cos x$



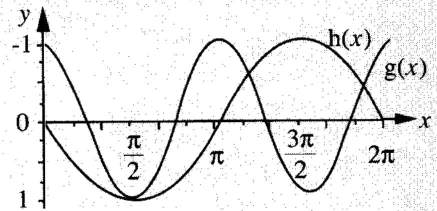
b. Sketch, on two separate pairs of axes, the other four functions.

3. The function $f{:}x \to \cos x$ can also be expressed as $f{:}x \to \sin\left(\frac{\pi}{2} - x\right)$. In a similar way, express each of the following in terms of sine:

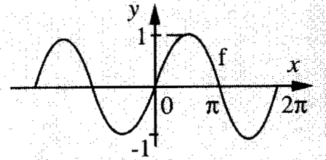
a. $\cos (x - \frac{\pi}{2})$ b. $\cos (x + \frac{\pi}{2})$ c. $\cos (x - \pi)$

d. $\cos (x + \pi)$ e. $-\cos x$

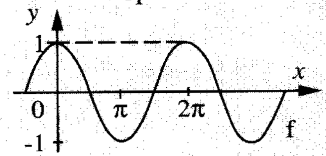
4. The diagram shows the graphs of two trigonometrical functions g and h.
   a. Write down the equation of g and the equation of h.
   b. What is the period of h?
   c. Using the graphs, or otherwise, find the value(s) of x for which $g(x) = h(x)$ in $0 \le x \le 2\pi$.



5. a. The function $f(x) = \sin x$ is shown in the diagram. Copy the diagram, and on it sketch and label the functions $g(x) = \sin 2x$ and $h(x) = 3 \sin x$.



   b. Copy the diagram of $f(x) = \cos x$, and on it sketch the functions: $m(x) = \cos (x - \frac{\pi}{2})$
   $n(x) = 2\cos x$



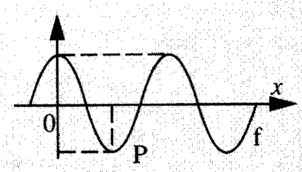
   c. Write $m(x)$ in terms of *sine* only.

6. The graph is of the function $y = A\cos Bx$. What are the co-ordinates of point P if:

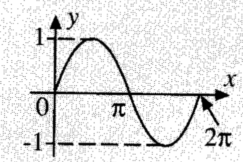
a. A = 2, B = 1? b. A = 3, B = 2?

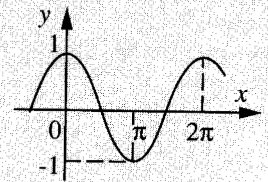
c. A = 1, B = 3? d. A = 4, B = 3?

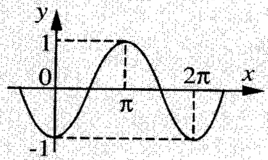
e. A = 2, B = 4?

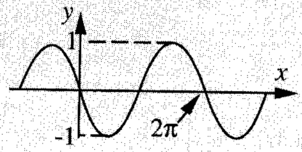


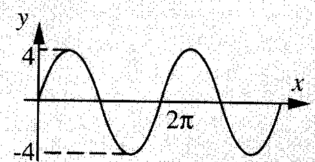
7. Write equations for the functions represented by the following graphs:

a. 

b. 

c. 

d. 

e. 

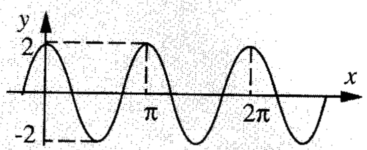
8. A mark on a steadily revolving ferris wheel is at a height h metres above the ground after t seconds. The height h is given by:

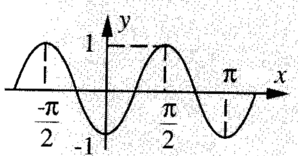
$$h = 8 + 7\sin\frac{\pi t}{12}$$

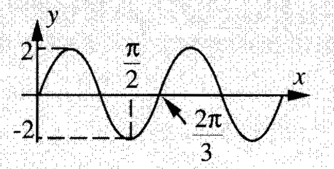
a. How high above the ground is the mark after 2 seconds?
b. After how many seconds is the mark at its greatest height? What is this height?

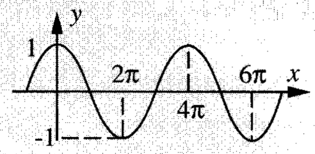
9. a. Sketch a graph of the function $s{:}x \to \sin\frac{x}{2}$.
b. Write down the period of function s.
c. What is the value of $\sin\frac{x}{2}$ when $x = \frac{\pi}{3}$?
d. Solve $\sin\frac{x}{2} = \frac{1}{2}$ for $0 \le x \le 2\pi$.

10. Write out the equations of the functions whose graphs are shown:

a. 

b. 

c. 

d. 

Problems and Investigations

1. Investigate the number of times the line $y = \frac{1}{n}x$ intersects the graph $y = \sin x$.

2. A mathematician analysing the motion of a small mark on a turning wheel believes the height in metres of the mark above the floor at time t in seconds is given by:

H = A sin kt + B, where k is a constant.

He records the following data:

t	0	0.3	0.5	0.7	1	1.3	1.5	2
H	8	10.4	11	10.4	8	5.6	5	8

Investigate his belief.

3. Suppose the mathematician, from question 2, had come up with this data:

t	0.1	0.2	0.9	1.3	1.7	1.8	2
h	6.16	6.31	7.21	7.47	7.47	7.43	7.30

What conclusions could be reached now?

27. ARC LENGTH AND SECTOR AREA

ACHIEVEMENT OBJECTIVES

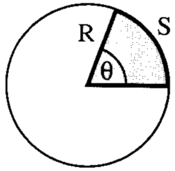
On completion of this chapter, students should be able, at:

LEVEL 7 MEASUREMENT

- *to model a given situation, using trigonometry (including radian measure) to find and interpret measures in context, and evaluate their findings*

Introduction

The length of an **arc** and the area of a **sector** are found by using radian measure. In the following diagram of a circle of radius R, the sector is shown shaded, θ is the size of the angle in radians, and S is the length of the arc. The arc, S, is said to **subtend** the angle θ.



When $\theta = 2\pi$, S is the circumference C:

$\therefore \frac{S}{C} = \frac{\theta}{2\pi}$	[the length of S is proportional to the size of θ]
$\therefore S = \frac{\theta}{2\pi} \times C$	[multiplying by C]
$\therefore S = \frac{\theta}{2\pi} \times 2\pi R$	[since $C = 2\pi R$]
$\therefore S = R\theta$	[cancelling 2π]

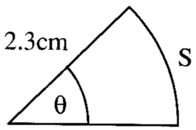
The formula for the length of an arc, together with the similarly derived formula for the area A of the sector, is shown below where the angle θ is in radians:

$$A = \frac{1}{2}R^2\theta$$
$$S = R\theta$$

Worked Examples

If the angle is given in degrees it must be changed to radians before using the formulae $A = \frac{1}{2}R^2\theta$ or $S = R\theta$.

Example A: In the sector shown, θ is initially 0.8 radians.
a. Find the length of the circular arc.
b. Find the area of the sector if θ changes to 70°.

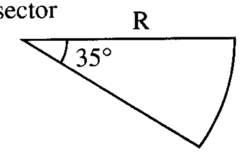


Solution:

a. $S = R\theta$
$= 2.3 \times 0.8$
$= 1.8\text{cm}$ (1 dp)

b. $70° = \frac{70}{180} \times \pi$ radians
$= 1.22$ radians
$\therefore$ area $= \frac{1}{2}R^2\theta$
$= \frac{1}{2} \times 2.3^2 \times 1.22$
$= 3.2\text{ cm}^2$ (1 dp)

Example B: An arc subtends an angle of 35° forming a sector of area 24.2 cm^2. Find the length of the radius of this arc.



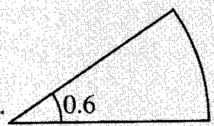
Solution: In the diagram the radius R is shown.

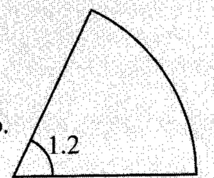
$35° = \frac{35}{180} \times \pi$ radians	[changing 35° to radians]
$A = \frac{1}{2}R^2\theta$	
$24.2 = \frac{1}{2} \times R^2 \times \frac{35 \times \pi}{180}$	[substituting for A and θ]
$R^2 = \frac{2 \times 180 \times 24.2}{35 \times \pi}$	[making R^2 the subject]
$R^2 = 79.23187796$	
$R = 8.9\text{cm}$ (1 dp)	

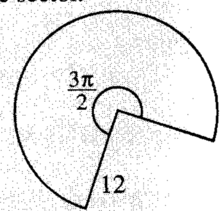
Note: To avoid the build-up of errors, rounding off is left until the final step.

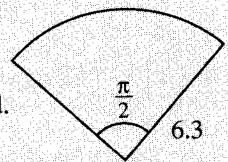
Exercise 27a

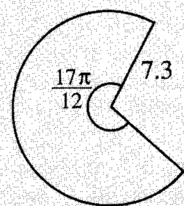
1. In each diagram the marked angles are in radians and the lengths are in metres. Find: i. the length of the arc. ii. the area of the sector.

a.


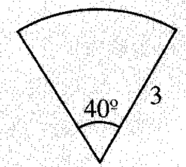
b.


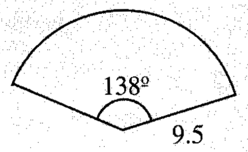
c.


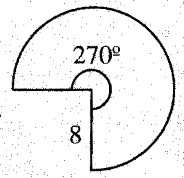
d.


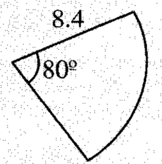
e.


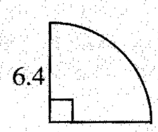
2. For each of the following diagrams find:
   i. the length of the circular arc. ii. the area of the sector.

a. 

b. 

c. 

d. 

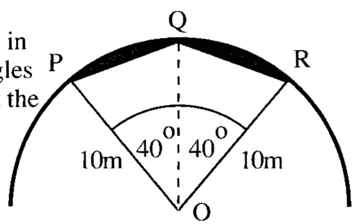
e. 

3. a. An arc 5 cm long is on a circle of radius 7 cm. What angle, in radians, does it subtend?
   b. An arc 8 cm long is on a circle of radius 12 cm. What angle, in degrees, does it subtend?
   c. A sector of a circle is bound by an arc 4 cm long and radius of 3 cm. What is the angle of the sector?
   d. How long is the arc which subtends an angle of 40° in a circle, radius 5 m?
   e. A sector of a circle with radius 4 cm has an angle of 50°. What is its area?

4. a. Find the angle in degrees of the sector of a circle with area 18 cm^2 and radius 7 cm.
   b. A sector of a circle subtends an angle of 36° and is bound by an arc and two radii. The arc is 7 cm long. Find the area of the sector.
   c. An arc subtends an angle of 72° and covers an area 60 cm^2. Find the length of the radius.
   d. An arc subtends an angle of 48° and makes a sector with area 54 cm^2. Find the length of the arc.
   e. A sector of a circle has a perimeter of 70 cm. The radius of the circle is 20 cm. Find the area of the sector.

Problems Involving Trigonometry

Example C: The kite OPQR is shown below in the sector OPR. The sides PQ and QR of triangles OPQ and OQR each subtend an angle of 40° at the centre. The circle has a radius of 10 m. Find:
a. the area of the kite
b. the shaded area.



Solution:

a. For each triangle:

$$\text{area} = \tfrac{1}{2}ab\sin C$$
$$= \tfrac{1}{2} \times 10 \times 10 \times \sin 40° \qquad \text{[substituting]}$$
$$= 32.14 \text{ m}^2 \qquad \text{[leave to 2 dp accuracy]}$$

area of the kite = area OPQ + area OQR
$= 64.28 \text{ m}^2$ [area OPQ = area OQR = 32.14]
$= 64.3 \text{ m}^2$ (1 dp)

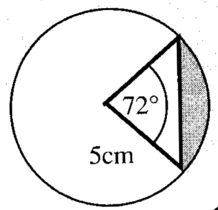
b. The required area = area of sector OPR – area of kite OPQR

$$\text{area sector OPR} = \tfrac{1}{2}R^2\theta$$
$$= \tfrac{1}{2} \times 10^2 \times \tfrac{80}{180} \times \pi \qquad \text{[changing to radians \& substituting]}$$
$$= 69.8 \text{ m}^2 \text{ (1 dp)}$$

∴ area required is 69.8 – 64.3 = 5.5 m^2

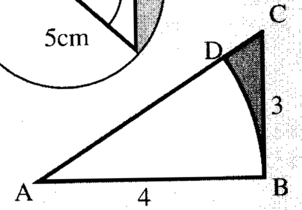
Exercise 27b

1. Find the area of the shaded region in the diagram opposite.

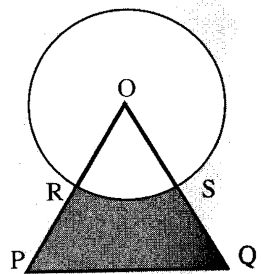


2. In this diagram find:
   a. the area of the shape CDB which is shaded.
   b. the perimeter of CDB.

   [Note: ABD is a sector of a circle of radius 4.]



3. In this diagram the triangle OPQ is an equilateral triangle with sides 2 m long. The circle has a radius of 1 m.
   a. Find the area of the shaded figure RSQP.
   b. Find the perimeter of RSQP.



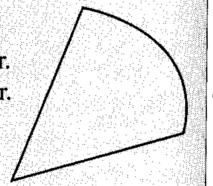
4. A regular hexagon with sides 2 m long is inscribed inside a circle. Find the area of that part of the circle which lies outside the hexagon.

5. A circle is drawn within a square with sides 3 m long. Each side is a tangent to the circle. Find the area of that part of the square not contained within the circle.

6. Two circles of radius 4 m and 5 m are placed so their centres are 6 m apart. Find the area of the region common to both circles.

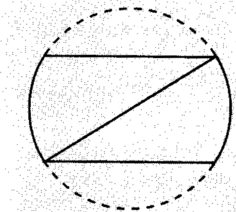
Problems and Investigations

1. A wire of length 10 is bent into the shape of a circular sector. Investigate the problem of maximising the area of this sector.



2. Find a formula for the area of a regular n-gon inscribed in a circle of radius 1.

3. Investigate the following conjecture: It is possible to find two congruent sectors which lie on top of each other as in the diagram so that the curved parts of each are part of the circumference of the same circle.



28. ANTIDIFFERENTIATION (INTEGRATION)

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

- LEVEL 7 MEASUREMENT AND CALCULUS
  - *to use integration and antidifferentiation in real and simulated situations*

Introduction

Antidifferentiation, as the name suggests, is the opposite process to differentiation. Antidifferentiation is often called **integration.** When an expression is *antidifferentiated* or *integrated*, the result is called the **antiderivative** or **integral**.

Example A: Find the antiderivative of $3x^2$.

Solution: If x^3, $x^3 + 1$ or $x^3 - 7$ are differentiated, $3x^2$ results. Generally if $x^3 + C$, where C is any fixed number, is differentiated, $3x^2$ results. Thus the antiderivative of $3x^2$ is $x^3 + C$, where C is called the **constant of integration**.

Example B: The antiderivative of x^5 is $\frac{x^6}{6} + C$, because if $\frac{x^6}{6} + C$ is differentiated, x^5 results.

The examples above can be generalised to give the following result:

The antiderivative of x^n is $\frac{x^{n+1}}{n+1} + C$.

Note: The result may be checked by differentiating $\frac{x^{n+1}}{n+1} + C$ and confirming that the derivative is x^n.

Example C: The antiderivative of x^{12} is $\frac{x^{13}}{13} + C$.
The integral of x^{-9} is $-\frac{x^{-8}}{8} + C$.

Further Rules for Antidifferentiation

The *sum* or *difference* of the two functions, *f* and *g*, may be antidifferentiated by antidifferentiating each function separately.

Example D: The antiderivative of $x^3 - x^2$ is the antiderivative of x^3 minus the antiderivative of x^2. The antiderivative of x^3 is $\frac{x^4}{4} + C_1$ and the antiderivative of x^2 is $\frac{x^3}{3} + C_2$ where C_1 and C_2 are the constants of integration.
The antiderivative of $x^3 - x^2$ is written $\frac{x^4}{4} - \frac{x^3}{3} + C$ because the difference between the two constants C_1 and C_2 is another constant, C.

Expressions of the type ax^n can be antidifferentiated by antidifferentiating x^n and then multiplying by a.

Example E: The antiderivative of $4x^8$ is the antiderivative of x^8 multiplied by 4, which is $\frac{4x^9}{9} + C$.

Notation

Integrating a function means finding its antiderivative or integral. The symbol for the general integral, antiderivative, or integral of the function f is $\int f(x)\,dx$.

Example F: a. $\int 5x^5\,dx = \frac{5}{6}x^6 + C$

b. $\int \frac{4}{x^2}\,dx = \int 4x^{-2}\,dx$ [rearranging to the form ax^n]

$= \frac{4x^{-1}}{-1} + C$ [integrating]

$= -4x^{-1} + C$ or $\frac{-4}{x} + C$

c. $\int \sqrt[3]{x^2}\,dx = \int x^{\frac{2}{3}}\,dx$ [rearranging]

$= \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C$ [integrating]

$= \frac{3}{5}x^{\frac{5}{3}} + C$

Note: Strictly, $\int f(x)\,dx$ is read 'the general integral of f *with respect to x*'. Usually this is unnecessary. However, if there is more than one letter involved, such as when one or more letters are constant, it is important to say which letter is the variable by using the full and correct expression.

Example G: Integrate yx with respect to x, i.e., find $\int yx\,dx$.

Solution: $\int yx\,dx = \frac{yx^2}{2} + c$

Note: By comparison, if yx is integrated *with respect to y*, we get $\frac{y^2x}{2} + C$.
i.e. $\int yx\,dy = \frac{y^2x}{2} + C$

Sometimes expressions need to be **expanded** before they can be **integrated.**

Example H: $\int (x^2 + 1)(x + 2)dx = \int (x^3 + 2x^2 + x + 2)dx$ [expanding]

$= \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + 2x + c$

Use of Calculators

There are a few advanced calculators such as the Hewlett Packard HP 48G which enable the user to work out integrals. Owners should consult their manuals.

Exercise 28

1. Antidifferentiate each of the following functions, with respect to x:

a. x b. x^3 c. $2x^4$ d. $x^{11} + x^{12}$
e. $x^6 - x^5 + x^4$ f. $2x^7 + 3x^6$ g. $4x^3 - 2x^5$ h. $2x^3 - 3x^4$
i. $x(x-3)$ j. 2 k. 3 l. $x - 3$
m. $x^2 - 3x + 4$ n. $(x+3)(x+4)$ o. $(2x+3)^2$
p. mx^2 q. $(2x+3)(3x^2+4)$ r. $(2x^2+5)^2$
s. a^2 t. $\frac{x}{b}$ u. $cx + \frac{x^2}{d}$
v. $Ax^2 + Bx + C$ w. $\frac{x^2}{A} + \frac{x}{B} + \frac{1}{C}$ x. $(Ax^2 + Bx + C)^2$
y. $(Ax + x^2)^2$

2. Integrate the following functions with respect to x:

a. $3x$ b. $3x^4$ c. $3x^7$ d. $2x^8$
e. $x^{13} - x^{10}$ f. $2x^7 + 8x^5$ g. $x^4 - 4x^3 + 3x$ h. $(x^2+1)(x+3)$
i. $(2x+5)^2$ j. $Ex + D$

3. Find the following:

a. $\int 4x\,dx$ b. $\int \frac{1}{3}x^2\,dx$ c. $\int 7x^3\,dx$ d. $\int 2r\,dr$
e. $\int 8r^8\,dr$ f. $\int 5t^6\,dt$ g. $\int 8\,dx$ h. $\int a\,dx$
i. $\int t^5\,dx$ j. $\int 2t^5\,dt$ k. $\int (2x+4)dx$
l. $\int (x^3 - 3x^2 + 5x + 2)\,dx$ m. $\int (4x+3)^2dx$
n. $\int (2x^2+1)^2dx$ o. $\int x^2(x+3)dx$

4. Find the following:

a. $\int x^{-3}\,dx$ b. $\int 4x^{-6}\,dx$ c. $\int \frac{1}{x^7}\,dx$ d. $\int \frac{1}{x^{10}}\,dx$
e. $\int \frac{3}{x^{12}}\,dx$ f. $\int \frac{5}{u^6}\,du$ g. $\int \frac{dv}{v^3}$ h. $\int \frac{3dx}{x^5}$

i. $\int \frac{x^{-3}}{4}\,dx$ j. $\int \frac{1}{5x^5}\,dx$ k. $\int \frac{1}{7u^3}\,du$ l. $\int \frac{2\,dx}{9x^7}$

m. $\int \frac{3\,dx}{5x^4}$ n. $\int \left(\frac{1}{x^2}+\frac{1}{x}\right)^2 dx$ o. $\int \left(\frac{2}{u}-\frac{1}{3u^2}\right)^2 du$

5. Find the following:

a $\int \sqrt{x}\,dx$ b. $\int \sqrt[4]{x}\,dx$ c. $\int \sqrt[3]{x^4}\,dx$ d. $\int \sqrt[5]{t^7}\,dt$

e. $\int \sqrt[8]{u^7}\,du$ f. $\int \sqrt[8]{t^3}\,dt$ g. $\int 5\sqrt[3]{x^2}\,dx$ h. $\int 9\sqrt[3]{x^4}\,dx$

i. $\int 10\sqrt[4]{x^7}\,dx$ j. $\int \frac{3}{5}\sqrt{x^4}\,dx$ k. $\int \frac{7}{6}\sqrt{x^3}\,dx$ l. $\int \frac{dx}{\sqrt{x}}$

m. $\int \frac{dx}{\sqrt{x^3}}$ n. $\int \frac{dx}{\sqrt[3]{x^5}}$ o. $\int \frac{dx}{\sqrt[5]{x^4}}$ p. $\int \frac{3du}{\sqrt[4]{u^5}}$

q. $\int \frac{5\,dx}{6\sqrt[7]{x^3}}$ r. $\int (\sqrt{x+3})^2\,dx$

s. $\int (\sqrt[3]{x}+1)^2\,dx$ t. $\int \left(2\sqrt{u}+\frac{3}{\sqrt{u^5}}\right)^2 du$

29. FINDING THE CONSTANT OF INTEGRATION

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 MEASUREMENT AND CALCULUS

- *to use integration and differentiation in real and simulated situations*

Finding the Constant of Integration

With enough information, the **constant of integration** can be evaluated.

Example A: $f'(x) = 2x + 3$ and $f(2) = 5$. Find $f(x)$.

Solution: $f'(x) = 2x + 3$

$\therefore\ f(x) = x^2 + 3x + c$ [integrating]

$\therefore\ f(2) = 4 + 6 + c$ [substituting 2 for x in $x^2 + 3x + c$]

$\therefore\ 5 = 10 + c$ [since f(2) = 5]

$\therefore\ c = -5$

$\therefore\ f(x) = x^2 + 3x - 5$

Example B: $\frac{dy}{dx} = 3x^2 - x + 2$ and $y = 6$ when $x = 2$. Find an expression for y in terms of x.

Solution: $\frac{dy}{dx} = 3x^2 - x + 2$

$\therefore\ y = x^3 - \frac{1}{2}x^2 + 2x + c$ [integrating]

$\therefore\ 6 = 8 - 2 + 4 + c$ [substituting $y = 6$ and $x = 2$]

$\therefore\ c = -4$

$\therefore\ y = x^3 - \frac{1}{2}x^2 + 2x - 4$

Example C: The rate at which the volume changes with respect to time, t, is given by the expression 10 – 2t. The volume is initially 50. Find an expression for volume with respect to time.

Solution: Letting V be volume and using the information given:

$\frac{dV}{dt} = 10 - 2t$

$V = 10t - t^2 + c$ [integrating]

$50 = 0 - 0 + c$ [substituting V = 50 when t = 0]

$\therefore\ 50 = c$

$\therefore\ V = 10t - t^2 + 50$

Exercise 29a

1. $f'(x) = x$, and $f(0) = 0$. Find an expression for $f(x)$.
2. $g'(x) = x^2 + x$ and $g(0) = 2$. Find an expression for $g(x)$.
3. $\frac{dy}{dx} = 3x^2$ and $y = 4$ when $x = 1$. Find an expression for y in terms of x.
4. $\frac{dy}{dx} = 4x^3 + 2x$ and $y = 5$ when $x = 2$. Find an expression for y in terms of x.
5. $y' = 2x^3 + 1$ and $y = 7$ when $x = 2$. Find an expression for y in terms of x.
6. $f'(x) = 3x + 2$, $f(3) = 4$. Find an expression for $f(x)$.
7. $g'(x) = 5x^2$, $g(4) = 11$. Find an expression for $g(x)$.
8. $\frac{dy}{dx} = 2x^2 - x + 3$, $y = 16$ when $x = 6$. Find an equation for y in terms of x.
9. $\frac{dy}{dx} = 3x - x^2$, $y = 42$ when $x = 12$. Find an expression for y in terms of x.
10. $y' = 2x^2 - 3x + 2$. The point (1, 5) lies on the graph of this function. Find y in terms of x.
11. $y' = x + 3$. The point (2, 1) lies on the graph of this function. Find y in terms of x.
12. $\frac{dE}{dr} = -2r$. E is 5 when r is 4. Find the equation relating E to r.
13. a. A balloon has volume increasing at the rate of 2t + 1 cubic centimetres per second. Initially the volume is $12 cm^3$. Find the volume at time t, where t is in seconds.
    b. The balloon will burst when the volume is $210 cm^3$. Find the time when the balloon will burst.
14. a. The water in a tank is escaping at the rate of 60 + 2t cubic centimetres per second. Initially the volume was $11\ 000 cm^3$. Find an expression for the volume at time t.
    b. Find the time when the tank becomes empty.
15. a. The rate at which fuel is used by a vehicle after the first kilometre of travel is $5 + \frac{6}{x^2}$ litres per kilometre when it has travelled x km. After travelling for 1 km the vehicle has 150 litres of fuel. Find an expression for the amount of fuel it has when it has travelled x km.
    b. When will the volume of fuel be 100 litres?

Distance, Speed and Acceleration

If the position, s, of an object is expressed as a *function of time*, t, then the **velocity**, v, of the object is the *instantaneous rate of change of the distance* with respect to time and is written $v = \frac{ds}{dt}$. Similarly, the **acceleration**, a, is the *instantaneous rate of change of the object's velocity* with respect to time and is written $a = \frac{dv}{dt}$.

Integrating the expressions $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt}$ gives:

$$s = \int v\, dt \quad \text{and} \quad v = \int a\, dt$$

Note: Speed is the absolute value of velocity.

Example D: A car moves so that its distance in kilometres from a town is given by $s = 80t(2 - t)$ where t is time in hours.

a. Find the distance of the car from the town after 1.25 hours.
b. Find an expression for the velocity at time t.
c. Find the velocity of the car after 45 minutes.
d. Find when the car is at the greatest distance from the town and the magnitude of that distance.
e. Find the velocity of the car when t = 1.5 hours and explain why it has a negative value.
f. Show that the car has a constant acceleration and find it.

Solution:

a. The distance from the town when t = 1.25 is:

$s = 80 \times 1.25 \times (2 - 1.25)$ [substituting t = 1.25 into $s = 80t(2 - t)$]

$= 75$ km

b. The car's velocity is $v = \frac{ds}{dt}$

$\therefore\ v = \frac{d}{dt}(80t(2 - t))$ [substituting 80t (2 – t) for s]

$= \frac{d}{dt}(160t - 80t^2)$ [expanding the bracket]

$= 160 - 160t$

c. The velocity when t = 45 minutes is found by substituting in $v = 160 - 160t$:

$v = 160 - 160 \times 0.75$ [since 45 minutes = 0.75 hours]

$= 40$ km per hour

d. The greatest distance of the car from the town will occur when $\frac{ds}{dt} = 0$.

$\therefore\ 160 - 160t = 0$

$\therefore\ t = 1$ hour [solving the equation]

$\therefore$ the maximum distance $= 80 \times 1 \times (2 - 1)$ [substituting in 80t (2 – t)]

$= 80$ km

e. The velocity when t = 1.5 hours is:

$v = 160 - 160 \times 1.5$ [substituting t = 1.5]

$= -80$

The *negative* sign shows that the car is moving back towards the town.

f. The acceleration of the car is given by:

$$a = \frac{dv}{dt}$$
$$= \frac{d}{dt}(160 - 160t)$$
$$= -160 \text{ km (per hour)}^2, \text{ which is constant.}$$

Example E: A toy car moves with velocity $16t - 2t^2$ cm per second for 8 seconds. Initially it is 5 cm away from its base.
a. Find the velocity when 3 seconds have elapsed.
b. Find an expression for the acceleration at time t.
c. Find an expression for the distance of the toy from its base at time t.

Solution: Letting s be the distance, v the velocity, and a the acceleration gives:
a. After 3 seconds $v = 16 \times 3 - 2 \times 3^2$ [substituting t = 3]
$= 30$ cm per second

b. $a = \frac{dv}{dt}$
$= \frac{d}{dt}[16t - 2t^2]$ [since $v = 16t - 2t^2$]
$= 16 - 4t$ [differentiating]

c. $s = \int v \, dt$
$\therefore s = \int (16t - 2t^2) \, dt$ [substituting $v = 16t - 2t^2$]
$\therefore s = 8t^2 - \frac{2t^3}{3} + c$

Initially when t = 0, s = 5 hence:
$5 = 0 - 0 + c$ [substituting into $s = 8t^2 - \frac{2t^3}{3} + c$]
$\therefore s = 8t - \frac{2t^3}{3} + 5$

Exercise 29b: Distance, Velocity and Acceleration

1. A man moves so that his distance from his house after t minutes is $50t - 5t^2$ metres.
   a. How far is he after 2 minutes? b. How far is he after 4 minutes?
   c. How far is he after 8 minutes? d. When does he return to his house?
   e. What is his velocity after t minutes? f. What is his velocity after 3 minutes?
   g. What is the greatest distance he is away from his house?
   h. What is his acceleration after time t?
   i. What is his acceleration after 3 minutes?

2. A stone is projected vertically upwards with a speed of 30 metres per second. Its height after t seconds is given by $h = 30t - 5t^2$.
   a. How high is it after 2 seconds?
   b. How high is it after 1 second?
   c. How long does it take to return to the ground?
   d. What is the velocity after t seconds?
   e. What is the velocity after 4 seconds?
   f. What is the velocity when it returns?
   g. What is the greatest height it reaches?
   h. What is the acceleration after time t?

3. A rocket is moving from an observation point with a velocity given by $100 - 4t$ metres per second.
   a. What is the velocity after 5 seconds? b. When does the rocket come to rest?
   c. When time is zero the rocket is 100 m from the observation point. Find the distance of the rocket from the observation point when time is t.
   d. Find the distance the rocket is away from the observation point when 3 seconds have gone by.
   e. What is the acceleration of the rocket?

4. A toy moves with a velocity given by $50 - 3t$ cms per second. Initially it is 10 cm from its operator.
   a. Find a formula for the distance from the operator after t seconds.
   b. Find the distance from the operator after 4 seconds.
   c. Find the distance from the operator after 6 seconds.
   d. When does the toy come to rest?
   e. What is the acceleration of the toy?

5. A toy initially has a velocity of 5 cm per second. It is subject to an acceleration of 3 cm per second^2.
   a. What is its velocity after t seconds?
   b. What is its velocity after 10 seconds?
   c. When is its speed 45 cm per second?

6. The distance s of an object from an observer is given by $s = 3t^2 - t^3$.
   a. Find the velocity at time t. b. Find the acceleration at time t.
   c. What is the velocity when t = 2? d. What is the acceleration when t = 1.
   e. Solve the equation s (t) = 0.

7. $v = 2t - t^2$ describes the velocity in metres per second of an object from an observer who initially held it.
   a. How far away is it after 1 second?
   b. What is the velocity at time t = 3?
   c. When does it come to rest?
   d. What is its maximum distance from the observer in the first 3 seconds?

8. A projectile is fired vertically upwards so that its velocity v (in metres per second) after t seconds is given by $v = 40 - 10t$.
   a. Find its initial velocity (i.e. v when $t = 0$).
   b. Find its acceleration.
   c. Show carefully that, after t seconds, its height h (in metres) above the point of projection is given by $h = 40t - 5t^2$. Begin with $v = 40 - 10t$.
   d. Find the time it takes to reach its greatest height.
   e. Find its greatest height above the point of projection.
   f. State the time it takes to return to its point of projection.

9. The vertical component of the velocity of a hot air balloon, v metres per second, after it is released from the ground, is given by the relation $v = 2 - \frac{1}{10}t$.
   a. What vertical distance does the balloon rise in the first two seconds?
   b. What is the vertical acceleration of the balloon?
   c. How long does it take for the balloon to return to the ground?

10. A vehicle is moving with a speed of 4 metres per second when the brakes are gradually applied resulting in a deceleration of - 0.5t metres per second^2 (t is time in seconds). Find how far the car travels before coming to rest.

Problems and Investigations

1. A stone is thrown at an angle of 40° with a speed of 30 metres per second. The horizontal speed with which it is moving away from the thrower is 30 cos 40°. Investigate this problem in order to find the horizontal distance the stone thrown at 30 metres a second travels.

2. A vehicle is brought to rest by use of a brake which decelerates it at a constant rate. The following data is available:

x	44.35	188.75	241.15	280.35	306.35	314.4
t	1	5	7	9	11	12

   where x is the distance travelled after t seconds since the brake was applied. Find the deceleration.

30. DEFINITE INTEGRALS AND AREA

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 MEASUREMENT AND CALCULUS

- *to use integration and antidifferentiation in real and simulated situations*

Definite Integrals

The **definite integral** of f with respect to x between the limits of a and b is written $\int_a^b f(x)\,dx$. If F(x) is the **antiderivative** of f, then:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Definite integrals are calculated in the following way:

Example A:

$$\int_{-1}^{3} 2x\,dx = [x^2 + c]_{-1}^{3} \quad \text{[integrating } 2x\text{]}$$
$$= (3^2 + c) - ((-1)^2 + c) \quad \text{[substituting into } x^2 + c\text{]}$$
$$= 3^2 - (-1)^2 \quad \text{[simplifying]}$$
$$= 8$$

Note: The constant of integration always cancels out during the calculation. For this reason it is unnecessary to include it during the calculation as the above example shows.

Example B:

$$\int_4^9 \frac{dx}{\sqrt{x}} = \int_4^9 (x)^{-\frac{1}{2}}dx \quad \text{[since } \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}\text{]}$$
$$= \left[2(x)^{\frac{1}{2}}\right]_4^9 \quad \text{[integrating]}$$
$$= 2(9)^{\frac{1}{2}} - 2(4)^{\frac{1}{2}} \quad \text{[substituting into } 2(x)^{\frac{1}{2}}\text{]}$$
$$= (2 \times 3) - (2 \times 2)$$
$$= 2$$

Use of Calculators

There are a number of calculators which permit the quick calculation of definite integrals. An example is the Casio *fx*-7700 GB.
In order to calculate Example B the user would calculate $\int(1 \div \sqrt{x}, 4, 9)$.

Exercises 30a

1. Calculate the following definite integrals:

a. $\int_2^4 x\,dx$ b. $\int_1^5 x^2\,dx$ c. $\int_1^2 2t^3\,dt$ d. $\int_0^4 \frac{u^2}{5}\,du$

e. $\int_{-1}^1 (x^3 + 2x^2)dx$ f. $\int_2^3 2\,dx$ g. $\int_{-4}^2 3\,du$ h. $\int_{-3}^6 dx$

i. $\int_{-1}^2 x\,(2x + 1)\,dx$ j. $\int_{-2}^1 (u + 3)^2\,du$ k. $\int_{-3}^2 (r^3 - 2r^2 + 3r + 1)\,dr$

l. $\int_{-1}^{2\frac{1}{2}} x^2\,dx$ m. $\int_{\frac{1}{2}}^{1\frac{1}{2}} \frac{t^3}{4}\,dt$ n. $\int_{-3}^2 (p^2 + 2p - 3)\,dp$ o. $\int_{1.2}^{2.8} (r^2 - 2)\,dr$

2. Calculate the following definite integrals:

a. $\int_4^9 \sqrt{x}\,dx$ b. $\int_8^{27} \sqrt[3]{x}\,dx$ c. $\int_1^4 \sqrt{x^3}\,dx$ d. $\int_2^3 \frac{dx}{x^2}$ e. $\int_3^5 \frac{du}{u^3}$

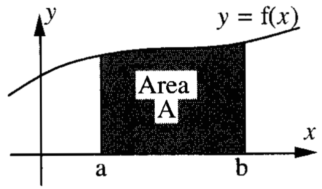
f. $\int_{1.2}^{3.6} \frac{du}{u^2}$ g. $\int_{0.6}^{1.8} (x)^{\frac{3}{5}}dx$ h. $\int_{1.57}^{2.48} \frac{3\,dx}{x^2}$ i. $\int_{1.97}^{4.82} \frac{du}{2u^3}$ j. $\int_{2.6}^{4.8} \frac{3\,dt}{5\sqrt{t}}$

Area

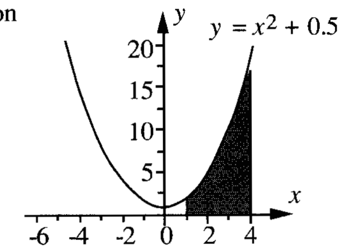
When the graph of a function is *above* the x axis, then the **area**, A, between the graph, the x axis and any two given values on the x axis is related to the definite integral of the function between these values.

The relationship is:

$$A = \int_a^b f(x)\,dx$$



Example C: Find the area of the shaded region in the diagram:



$$\text{Area} = \int_1^4 (x^2 + 0.5)\,dx$$

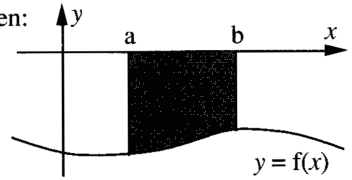
$$= \left[\frac{x^3}{3} + 0.5x\right]_1^4 \quad \text{[integrating]}$$

$$= \left(\frac{64}{3} + 2\right) - \left(\frac{1}{3} + \frac{1}{2}\right) \quad \text{[substituting into } \frac{x^3}{3} + 0.5x\text{]}$$

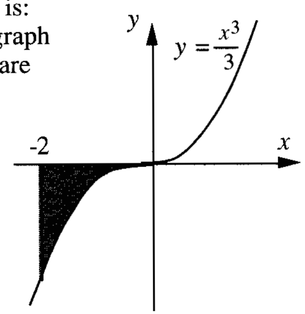
$$= 22\tfrac{1}{2}$$

If the graph has negative values, the definite integral is the *negative* of the shaded area. If A is the area then:

$$A = -\int_a^b f(x)dx \quad \text{or} \quad A = \left|\int_a^b f(x)\,dx\right|$$



Example D: The area of the shaded region shown is:

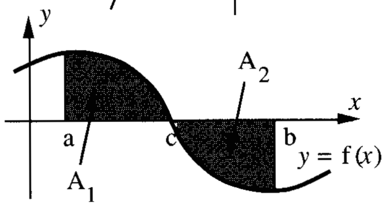


$$A = \left|\int_{-2}^0 \frac{x^3}{3}dx\right| \quad \text{[all points on the graph between -2 and 0 are below the } x \text{ axis]}$$

$$= \left|\left[\frac{x^4}{12}\right]_{-2}^0\right| \quad \text{[integrating]}$$

$$= \left|0 - \frac{4}{3}\right| = \frac{4}{3}$$

Note: It is advisable to sketch a graph before finding the area required.
When the function has both negative and positive values as shown, the shaded area under the graph is made up of an area above the axis, labelled A_1, and an area below the axis, labelled A_2.

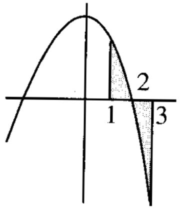


The shaded area is $A_1 + A_2 = \int_a^c f(x)\,dx + \left| \int_c^b f(x)\,dx \right|$

Example E: The area between $y = 4 - x^2$, $x = 1$ and $x = 3$ is

$$\int_1^2 (4 - x^2)dx + \left| \int_2^3 (4 - x^2)dx \right| = \left[4x - \frac{x^3}{3}\right]_1^2 + \left| \left[4x - \frac{x^3}{3}\right]_2^3 \right|$$

$$= \frac{5}{3} + \frac{7}{3} = 4$$

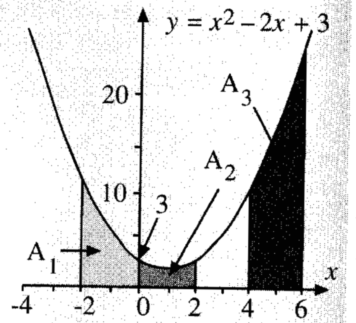


Use of Calculators

In order to find the area between $y = f(x)$, $x = a$ and $x = b$ using electronic calculators, the student should calculate $\int_a^b \text{abs}\,(f(x))\,dx$. The 'abs' function is the modulus or absolute value function.

Exercise 30b

1. The graph shown is a sketch of $y = x^2 - 2x + 3$. Find the values of A_1, A_2, A_3.



Find the area between:

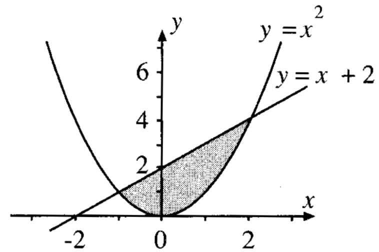
2. $y = x + 1, x = 1, x = 3$ and the x axis.
3. $y = 2x - 1, x = 2, x = 4$ and the x axis.
4. $y = x^2, x = 2, x = 5$ and the x axis.
5. $y = 4 - x^2, x = 1, x = 2$ and the x axis.
6. $y = 4x - x^2$ and the x axis.
7. $y = x^2 - 3x - 4$ and the x axis.
8. $y = x^3, x = -1, x = 2$ and the x axis.
9. $x = 1, y = (x - 2)(x - 4), x = 3$ and the x axis.
10. $x = 0, y = 4 - x^2, x = 3$ and the x axis.

Finding the Area between Curves

Example F: Find the area between $y = x + 2$ and $y = x^2$.

Solution: The graphs intersect at $x = -1$ and $x = 2$ [solving simultaneously].
The area required is shaded and is:

(area under $y = x + 2$) – (area under $y = x^2$) $= \int_{-1}^{2} (x + 2)\,dx - \int_{-1}^{2} x^2\,dx$



$$= \left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} + -2\right)\right] - \left(\frac{2^3}{3} - \frac{(-1)^3}{3}\right)$$

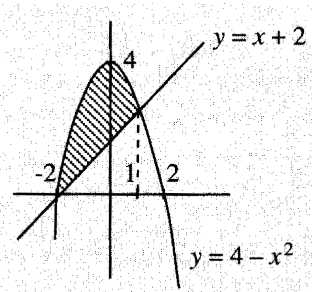
$$= 4\tfrac{1}{2}$$

Note: An alternative solution involves calculating the integral:

$$\int_{-1}^{2} \left[(x + 2) - (x^2)\right] dx$$

Exercise 30c

1. Find the shaded area:



2. Find the area between $y = 2x + 8$ and $y = 16 - x^2$. The two graphs intersect when $x = -4$ and $x = 2$.
3. Draw the graphs of $y = x^2$ and $y = x + 2$ then find the area enclosed between the two graphs.
4. Find the area between the graphs of $y = 4$ and $y = x^2$.
5. Find the area between $y = 8 - x^2$ and $y = x^2$.

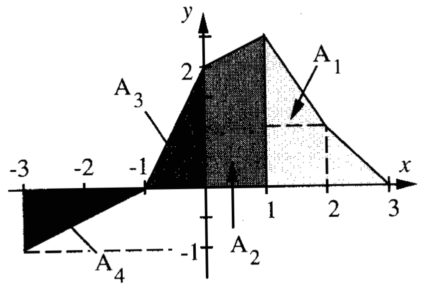
Using Areas to find Integrals

Previously in this chapter, integrals have been used to find areas. It is possible to reverse this process and, by using areas, find definite integrals.

Example G: For the function represented by the graph shown, find:

a. $\int_1^3 f(x)\,dx$ b. $\int_{-3}^{-1} f(x)\,dx$ c. $\int_{-3}^{3} f(x)\,dx$

Solution:



By finding areas the following results are obtained:

a. $\int_{1}^{3} f(x)\,dx$ is the area marked A_1 which is two triangles and a square.

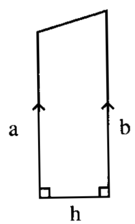
$$\text{Area} = \tfrac{1}{2} \times 1 \times 1\tfrac{1}{2} + 1 \times 1 + \tfrac{1}{2} \times 1 \times 1$$
$$= 2\tfrac{1}{4}$$

b. $\int_{-3}^{-1} f(x)\,dx$ is the negative of the area marked A_4 and is:

$$= -\left(\tfrac{1}{2} \times 2 \times 1\right) \text{ [area of a triangle]}$$
$$= -1$$

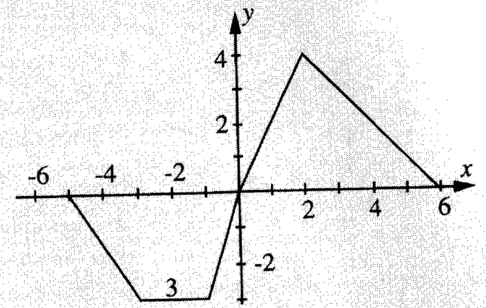
c. $$\int_{-3}^{3} f(x)\,dx = \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{3} f(x)dx$$
$$= -A_4 + A_3 + A_2 + A_1$$
$$= -1 + \tfrac{1}{2}\times 1 \times 2 + \tfrac{1}{2} \times 1\left(2 + 2\tfrac{1}{2}\right) + 2\tfrac{1}{4}$$
$$= -1 + 1 + 2\tfrac{1}{4} + 2\tfrac{1}{4} = 4\tfrac{1}{2}$$

Note: The area of A_2 is the area of a **trapezium**, i.e, $\frac{1}{2}(a + b)h$ where a, b are the lengths of the parallel sides and h is the distance between them.
A_1 in the above example could be treated as a trapezium plus a triangle.



Exercise 30d

1. Use the graph to evaluate the following definite integrals:



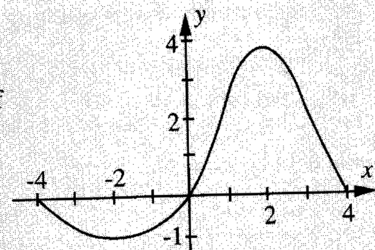
a. $\int_{0}^{2} f(x)\,dx$ b. $\int_{1}^{3} f(x)\,dx$

c. $\int_{-1}^{1} f(x)\,dx$ d. $\int_{-2}^{1} f(x)\,dx$

e. $\int_{-3}^{2} f(x)\,dx$ f. $\int_{-2\frac{1}{2}}^{2} f(x)\,dx$

g. $\int_{-2}^{3} f(x)\,dx$ h. $\int_{0}^{1\frac{1}{2}} f(x)\,dx$ i. $\int_{-2}^{0} f(x)\,dx$ j. $\int_{-3}^{3} f(x)\,dx$

2. Using the graph shown of the function $y = g(x)$ where g is defined on the set $-4 \le x \le 4$, find the approximate values of the following definite integrals:

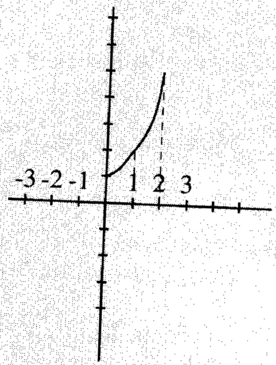


a. $\int_{3}^{4} g(x)\,dx$ b. $\int_{2}^{3} g(x)\,dx$

c. $\int_{2}^{4} g(x)\,dx$ d. $\int_{0}^{3} g(x)\,dx$ e. $\int_{-2}^{2} g(x)\,dx$ f. $\int_{-2}^{3} g(x)\,dx$

g. $\int_{-3}^{1} g(x)\,dx$ h. $\int_{-4}^{3} g(x)\,dx$ i. $\int_{-4}^{0} g(x)\,dx$ j. $\int_{-4}^{-3} g(x)\,dx$

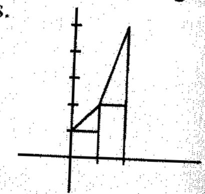
Problems and Investigations

1. 

The diagram shows the graph $y = x^2 + 1$ for $0 \le x \le 2$.

a. Find the area under $y = x^2 + 1$ exactly by calculating the integral $\int_0^2 (x^2 + 1)\,dx$.

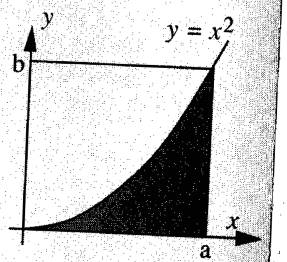
b. Find an approximation to the area by treating the curve joining (0, 1) and (1, 2) as straight and the curve joining (1, 2) to (2, 5) as straight, then working out the area using triangles and rectangles (see diagram below), or trapeziums.

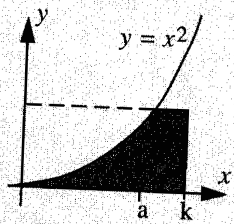


2 a. Draw the graph of $y = 9 - x^2$ for $0 \le x \le 3$ and find the area under this graph using calculus.

b. Find an approximate area by plotting the points (0, 9), (1, 8), (2, 5), (3, 0) on the graph, then using the method involving triangles and rectangles, or trapeziums, used in question 1b.

3. For the diagram to the right, find what fraction of the rectangle is shaded.



4. 

Find the value of k so that the shaded area is half the area of the rectangle in the diagram to the left:

5. Find the value of k so that the area between the curve $y = k - x^2$ and the x axis is 2.

6. Find the value of k so that the area contained between $y = x^k$, $x = 0$, $x = 1$ and the x axis is 3.

31. STATISTICS

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 6 STATISTICS

- *to formulate statistical questions about situations involving possible relationships between variables*
- *to formulate questions about variations over time in continuous processes*

LEVEL 7 STATISTICS

- *to plan a statistical investigation to make inferences about a population or experimental situation*

Introduction

Statistics is the collection, display and analysis of data.

Data means the numbers and information involved with and arising from an investigation.

The main *use* for statistics is to provide *evidence for decision making*. By using statistical techniques it is possible, on the basis of an investigation of only a fraction of a population (called a **sample**), to reach conclusions about the whole population. These conclusions have a good chance of being correct.

Statistics is an important branch of mathematics. It has widespread applications in science, commerce and industry.

In other fields of mathematics such as algebra, calculus and geometry, it is possible to achieve answers which are definitely right or wrong. In statistics, it is possible to reach conclusions which are wrong, even when the techniques employed have been used correctly. Furthermore, it is possible, by the use of statistical techniques, to draw completely false conclusions from data which is false, incomplete or 'biased'.

An example is the ever-changing assessment of the effects of a nuclear war. As ever more accurate data becomes available, correct statistical analysis reveals different consequences, which are often more and more serious.

Statistical Questions

The most important part and the first step in the process of statistics is the asking of a question(s). This provides a guide as to what comes next, usually an investigation to try and answer the question.

Example A: An orchard owner wants to know if the fruit packing is being done efficiently or whether it would be better to buy a new machine. The orchardist decides to take a sample of packed boxes and count the number of bruised fruit. From this sample, using statistical techniques, the proportion of *all* boxes with bruised fruit can be accurately estimated.

In Example A the orchard owner would have asked the questions:

Are there better ways of picking the fruit:
i. which allow more fruit to be packed without damage;
ii. whose cost is more affordable.

Having done this the orchard owner designed and carried out an investigation as outlined in Example A.

Exercise 31a

For each of the following situations formulate a statistical question and outline a possible way in which you might answer the question.

1. A worker in a cafeteria notices that some customers order tea, others coffee, others soft drink.
2. A passenger on the bus notices where people sit when they get on.
3. A car owner notices that the car does not seem to be using as much petrol after its recent servicing.
4. A coach shows the film 'Rambo' to the team before Saturday's game. The team wins by a big score.
5. A holidaying couple notice that their joints are stiff after their return home. During their holiday they frequently swam in a hot thermal pool.
6. A farmer believes that chickens which are given one type of feed produce larger eggs than chickens which are given another.
7. A shopper has the choice of going to two supermarkets from different chains which are equally convenient to shop at.
8. A television watcher notices the products advertised on different stations and the length of time for each commercial.
9. A radio listener believes his favourite station plays the same songs each day.
10. An economist claims that products which are heavily advertised cost more than similar products which are not advertised.
11. After repeating a physical movement the pulse increases.
12. After repeating an exercise the muscles used hurt.
13. After a period of time the end of a metal bar will get hot if the other end is heated.
14. The height of the water in a bay is different depending on the time of day.
15. The speed at which salt dissolves in water depends on the temperature of the water.

Surveys

A **survey** is a statistical investigation in which data are collected.

Surveys are usually carried out for the purpose of assisting decision making. Ideally, they are conducted in such a way that the results obtained are as close to reality as possible. Decision makers draw conclusions from these results and make their decisions based on these conclusions.

Sometimes decision makers have a definite idea about what they want to do and are interested only in the results of surveys which strengthen the case for their predetermined course of action. In such circumstances it is necessary for a statistician to try and design a survey which will produce the required result, while still looking plausible to as many people as possible.

Target Population

A **target population** is the population being investigated in a survey.

The target population should be carefully defined.

Example B:

a. It may be necessary to find out as much as possible about the ages of women when they get married. The target population is 'the ages at which women marry', rather then just 'women' or 'people'.
b. Similarly it may be necessary to find out the distances tyres can travel before becoming unsafe. The target population is not 'tyres', but 'distances at which tyres become unsafe'.

Exercise 31b

For each of the following identify the target population.

1. A catering company took a survey of high schools to find out the dates of their 'sports days'.
2. A market research company did a survey on brands of soap people use.
3. A teacher recorded the subjects taken by the pupils in a F7 class.
4. A zoologist catches mice and records their age.
5. A student surveys houses in different streets to determine the main colours on the exterior.
6. A researcher asks people about their attitudes to a number of different politicians.
7. A supermarket asks its customers to fill in a form giving their address so that it can work out how far its customers travel.
8. A toy company pays a school for the phone numbers of its F1 students so that it can find out more about the favourite toys of eleven and twelve year olds.

9. A magazine asks its readers to fill in a form which gives their name and address. On the form the customer have to put one of the numbers 0, 1, . . . , 10 indicating their attitude to new government policy with 0 being 'very low opinion' and 10 'very high opinion'. The magazine is keen to find out what its readers think about the policy. All readers who respond go into a draw for a trip overseas.
10. A petrol station records the number of passengers in the cars of customers.

Two Types of Survey

A. Census

A **census** is a survey where *every* member of the target population is investigated.

In New Zealand, a census is taken every five years. A census is expensive and takes a long time to analyse and report on, despite the use of computers. However, the information it provides allows planners in many fields to decide policy.

B. Sample Surveys

A **sample survey** investigates only a subset of a target population.

Any survey which does not involve investigating every member of a target population may be called a sample survey.

The following two examples illustrate how sample surveys may be used.

Example C: A merchandising organisation may wish to decide on the effectiveness of a nationally televised advertisement. This advertisement will have been seen by many people. A sample survey of only a tiny proportion of viewers will reveal information which has a strong chance of accurately reflecting the effectiveness of the advertisement on the whole television audience.

Example D: A factory producing expensive machines will subject only a few to vigorous use until they break down, in order to work out how long a guarantee or warranty should last for all the machines.

Sampling is a powerful tool for obtaining information about an entire population. In many cases it is the only way in which a survey can be done. For example:
* when the target population is too large or
* the cost of surveying too high or
* the population is inaccessible.

A sample survey would be used to find information about a species of fish. Despite the fact that we catch relatively few fish, we can draw conclusions based on this sample which have a very high chance of correctly describing the entire population.

Market Research and Opinion Polls

A **poll** is a survey which seeks *opinions* from people.

Periodically, the results of **political polls** are presented in the media to show the relative popularity of the political parties.
Polls are taken all the time for many reasons, mostly commercial, in order to find out the attitudes of people to a variety of things. Such a procedure is called **market research.**

In preparing such surveys, much work has to be done. Consideration has to be given to:

a. The purpose of the survey and for whom it is being carried out.
b. Survey design.
c. Execution of the survey.
d. Quality Control.
e. Analysis.
f. Reporting.

a. The purpose of the survey
Various private and public bodies get research organisations to conduct surveys on their behalf in order to find out public attitudes to policies, products, events, etc. They usually approach the organisation and discuss with a researcher the nature of their problem. If a survey will assist in the solving of the problem, an appropriate survey will be designed.

b. Survey design
A questionnaire is prepared by people who are usually trained in the social sciences. The preparation of the questionnaire is a matter of the highest importance. Consideration has to be given to such matters as:

i. The target population.
ii. Type of sample required - there are a number of different sampling strategies, see chapter 33.
iii. The actual questions to be asked. These are critical to a successful survey. Much research, over many years, has gone into the *best types* of question and the *methods* of questioning.

All questions are tested before being actually used in the field. This pre-testing is referred to as **piloting.**

c. Execution of the survey
Most **interviewers** are part-time workers who work in their spare time on a contract basis - they have no special training in market research.
It is the responsibility of the research organisation to ensure their interviewers are thoroughly briefed on each survey before it takes place. Such matters as the peculiarities of the questionnaire, and the area and streets in which they are to conduct interviews must be dealt with.

d. Quality control
As with any process, researching must have quality control to ensure that the research process is carried out properly. The work of interviewers is viewed very seriously by the managers of research organisations. The supervisors must check every filled-in questionnaire for inconsistencies. They also personally check a fraction of each interviewer's respondents to ensure the survey was conducted properly.

e. Analysis
The information from the completed and checked questionnaires is fed into computers and analysed by using statistical methods.

f. Report
Once all analyses are complete a report is written on the survey and presented to the client for whom the survey was done.

Exercise 31c

Choose a topic you will investigate and draft a plan for this investigation incorporating as many ideas as possible from this chapter. Your plan should include:

i. what you intend to investigate.
ii. the questions you will try to answer.
iii. your experiment or other data-gathering exercise.
iv. your target population.
v. how large a sample will be and why.
vi. how much time and resources you are allowed.
vii. how you will analyse your data.
viii. how you will report your results.

32. QUESTIONS AND QUESTIONNAIRES

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 6 STATISTICS

- *to formulate statistical questions about situations involving possible relationships between variables*
- *to identify data collection methodology*
- *to collect bi-variate measurement and discrete number data*

LEVEL 7 STATISTICS

- *to design and justify data collection methods*
- *to collect data*

Introduction

A **questionnaire** is a paper consisting of a series of questions used to obtain the particular information required. The questions are prepared by people expert in such work. These people are usually highly trained in the social sciences which include such subjects as sociology, anthropology, psychology and geography. In addition, they must have a good knowledge of statistics.

The questions used in questionnaires can be either **structured** or **unstructured.**

Structured Questions

These are questions which are put in such a way that the answer will always fall into *predetermined* categories. Structured questions are found on such forms as passport applications, investment applications, examination entries etc.

Example A:

Note: The **respondent** is the person answering the questionnaire.

1. **Sex**
   Circle that which applies: M/F

2. **Number of children**
   List the number in each age group residing in this dwelling:

Age	Number
Less than 5	
5 – 10	
10 – 14	

3. **Age**
   Put a tick below the age group in which you belong:

Age	< 15	15 – 19	20 – 24	25 – 29	30 – 39	40 – 50	> 50

4. Circle the letter beside each photocopier you are familiar with:

Kamakusa	K
Ricoh	R
Nashua	N
Xerox	X
IBM	I

5. *Rank* from 1 to 10 the current importance of the following matters by circling the appropriate number. 1 is least, 10 is most important.

Crime	1	2	3	4	5	6	7	8	9	10
Defence	1	2	3	4	5	6	7	8	9	10
Education	1	2	3	4	5	6	7	8	9	10
Employment	1	2	3	4	5	6	7	8	9	10
Energy	1	2	3	4	5	6	7	8	9	10
Industrial Relations	1	2	3	4	5	6	7	8	9	10
Inflation	1	2	3	4	5	6	7	8	9	10
Nuclear Policies	1	2	3	4	5	6	7	8	9	10
Pollution	1	2	3	4	5	6	7	8	9	10
Race Relations	1	2	3	4	5	6	7	8	9	10

6. *Rating scales*
   This school is most suitable for pupils who are:

very dull	dull	average	bright	very bright

[Respondent ticks where he or she believes the school rates. If the respondent thought the school was run mainly for very dull people, a tick would be placed in the box beside "very dull people". If the respondent thought the school catered more for the bright than the dull the fourth box from left should be ticked].

7. *Top of head recall*
   A list of names is prepared, say for cars:

Cortina	1	2
Corolla	1	2
Escort	1	2
Sierra	1	2
Civic	1	2
City	1	2

The interviewer asks the respondent, without showing the list, to recall brands of cars. The first brand on the list which is recalled has a (1) circled. All others recalled have (2) circled. Any not recalled have neither (1) nor (2) circled.
For example, suppose in one interview the respondent recalled City first, then said Cortina and Sierra and none of the others, then the response sheet would appear:

Cortina	1	②
Corolla	1	2
Escort	1	2
Sierra	1	②
Civic	1	2
City	①	2

Unstructured Questions

These are questions in which the answers are not restricted to particular categories. In a school exam, an essay question is an unstructured question while a multi-choice question is definitely structured.

Asking and recording the answers for unstructured questions requires a skilled interviewer.

The *object* of unstructured questions is to get the respondent to give as much information as possible without being *prompted* (i.e., without suggesting an answer).

Example B:
Interviewer: "What do you think about the government policy on industrial relations?"
Respondent: "It is inadequate."

The response is not very revealing and a good interviewer will now try to obtain more information. If the interviewer was to say, "Do you mean that the government should take a firmer stance?", then they are *prompting* the respondent. The question is *leading* the respondent to an answer.

A much better question would be: "In *what way* is it inadequate?"

This question contains no idea which the respondent can 'latch onto' and use as his or her own.

The interviewer should keep on *probing* with questions until the respondent has made their first response quite plain.

The probing should continue until the respondent has given an answer such as:

> "The government should take much firmer action with the companies and the unions!"

At this point the interviewer may feel the respondent has completely clarified the initial response. However, if this is not the case, the interviewer could continue:

Interviewer: "What action do you think the government should take?"

The answer of the respondent will probably terminate this particular line of probing. If the respondent does not know what kind of action the government could take, she/he will probably say so. Otherwise, the answer given will probably be sufficient to finish this probe. Either way, the interviewer has received a far more satisfactory series of responses than just accepting the original response of: "It is inadequate."

Another example of probing is shown in the following interview, the purpose of which was to determine a respondent's knowledge of dog food.

Example C: Interviewer: "Tell me all the types of dog food you know of."
Respondent: "Woofo, Doggydear, Waggyboy."
Interviewer: "Any others?"
Respondent: "Super Pup."
Interviewer: "And?"
Respondent: "Fido."
Interviewer: "Can you think of any others?"
Respondent: "No."
Interviewer: "No others?"
Respondent: "Sorry."

In this interview, the interviewer probed until the respondent had revealed *every type* of dog food she/he knew.

Exercise 32

The topic of questionnaires is particularly well suited to projects. The following exercises are suitable as projects.

1. Prepare, using the type of structured questions in Example A (1) - (6), a questionnaire to assess your classmates' attitudes to something you are interested in. Below is an example of a questionnaire that could be used to assess their attitudes to maths.

QUESTIONNAIRE ON ATTITUDES TO MATHS

a. What is your sex? Circle that which applies: M / F

b. Do you take maths because:
(Tick those which apply)

	You have to
	You want to
	You enjoy it
	You need it as a qualification

c. In what group does your fifth form mark belong? (Tick that which applies)

0 – 9	
10 – 19	
20 – 29	
30 – 39	
40 – 49	
50 – 59	
60 – 69	
70 – 79	
80 – 89	
90 – 100	

d. How importantly do you rate the teacher in your understanding of maths this year:
(Tick that which applies)

very important	
important	
same as all years	
not as important	
unimportant	

e. How importantly do you rate the textbook in your understanding of maths this year:

very important	
important	
same as all years	
not as important	
unimportant	

f. If you were asked to rate maths against the other subjects you are studying this year, where would it rate in terms of being your favorite subject:

Number of subjects taken	Rating of maths against other subjects

g. Has your attitude to maths changed during the past years at school:

improved	
stayed the same	
deteriorated	

(tick that which applies)

h. How relevant do you believe the study of maths to be to your future likely career:

irrelevant	unimportant	of little importance	of some importance	important	highly important

i. Listed below are 10 topics studied during most sixth form maths courses.

Circle the '1' if you have studied this topic.
Circle the '2' if you find it difficult.
Circle the '3' if you did not understand it.
(you may circle more than one number)

Factorising	1	2	3
Equations	1	2	3
Sequences and Series	1	2	3
Circle Geometry	1	2	3
Straight Line	1	2	3
Graph Drawing	1	2	3
Logarithms	1	2	3
Exponents	1	2	3
Trigonometry	1	2	3
Differentiation	1	2	3

j. Using the list of topics given in question 1 i. above, as well as any others, rate the 10 most difficult maths topics you have studied this year in order of increasing difficulty, '1' being the easiest topic studied, '2' being the next easiest etc.

Topic	Difficulty Ranking
	1
	2
	3
	4
	5
	6
	7
	8
	9
	10

Problems and Investigations

Using a questionnaire such as this could reveal a wealth of information even if only given to twenty people or so in your own maths class.
Among the things that could readily be found would be:

i. Differences in attitude to maths between male and female students.
ii. Differences in attitude to maths between students who have done well in earlier work and those who have not.
iii. Differences in pupils' opinions about the importance of teachers and text books.
iv. Whether interest in maths tends to increase or decrease as pupils get older.
v. The topics pupils find the easiest and the most difficult.

A survey such as this could be modified to be applied to nearly any topic.

1. Prepare an interview using unstructured questions on some topic and give it to, say, ten pupils in your class selected by drawing names from a hat or similar. As an example, here is a possible questionnaire designed to assess someone's general knowledge of the motor industry:

    a. Name as many different types of motor cars on sale as possible. [maximum of 15]
    b. Name as many different types of trucks on sale as possible. [maximum of 5]
    c. Name as many different types of oil on sale as possible. [maximum of 5]
    d. Name as many different brands of tyre on sale as possible. [maximum of 5]
    e. Name as many different motor vehicle manufacturers as possible. [maximum of 5]
    f. Name as many different parts of a motor engine as possible. [maximum of 10]
    g. Name as many different petrol companies as possible. [maximum of 5]
    h. Name as many different parts of a car as possible which will concern a driver while sitting in the driver's seat. [maximum of 5]

33. SAMPLING

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

- LEVEL 7 STATISTICS
  - *to design and justify sample selection and data collection methods*

Introduction

A **sample** is a subset of a population which is selected and studied so that conclusions may be reached about the population as a whole.
In taking samples, care must be taken to avoid what statisticians call **bias**.

A survey is **biased** if the conclusions obtained from the survey do not accurately describe the characteristics of the whole population, when many such surveys are taken and their results are averaged.

Example A: A newspaper or magazine invites readers to fill in and return a questionnaire about a controversial issue so that public feeling on the issue can be assessed. The results of such a survey are usually very biased for the reasons that only the interested readers of that particular newspaper or magazine will reply.

Example B: In addition, if the readers of the publication in Example A are drawn mainly from a particular section of the population such as 'university educated females with liberal views', or 'rugby players', then it is highly likely that the survey will be biased.

Sampling Strategies

Statisticians use many different methods for collecting samples. In general, the method used must ensure that every member of the population being surveyed has the same chance of selection. If the sample fulfills this criterion, it is said to be **free of bias.**

a. In the **random selection** method every individual in the population is given a different number, then the required sample is obtained by using random number tables as the following example shows:

Example C: Select a sample of 3 people from the group: Shane, Dave, Mary, John, Rachel, Jessica, Trudy, Ian, Den, Tasha, Melanie and Jane.

Solution: Each person is given a number:

Shane	01	Trudy	07
Dave	02	Ian	08
Mary	03	Den	09
John	04	Tasha	10
Rachel	05	Melanie	11
Jessica	06	Jane	12

Random number tables are then used to find a set of three. Selecting from the following list of random numbers:
9003, 1250, 2876, 0586, 0584, 7400, 9873, 0128, 5315, 1840, 0410, 8918, 4905, 2986, 9907, 9117, 4292, 9366 ....

Starting from 9003 and using the last 2 digits we find the numbers 90**03**, 04**10** and 49**05** are the first three which correspond to people in the group. Thus Mary, Tasha and Rachel are chosen.

Note:

a. If the same number appeared twice it would be disregarded after being used once.
b. The above result, though not biased, does not reflect the population because it suggests that the population is completely female when in fact half are males.
c. This method would be impractical with large populations.
d. It is usual to try to find your starting point in the random number tables by "using a pin". This means picking a starting point through some random procedure such as dropping a pin on the random number table and starting from where it lands.

b. The **stratified sampling** method avoids some of the problems of completely random sampling. In this method the population is split into **strata** which are subsets of the population which we might be interested in. In the previous example we might split the population into the strata: {males}, {females}, and then randomly sample from each stratum numbers in proportion to their numbers in the population. A further example follows:

Example D: A school has 300 pupils in Form 3; 300 in Form 4; 250 in Form 5; 100 in Form 6 and 50 in Form 7. A sample of 40 pupils is to be interviewed. The sample is to be chosen using strata given by form level. The total school population is 1 000.
The number of third formers in the sample is the proportion of third formers × number required = $\frac{300}{1000} \times 40 = 12$.
Similarly there are 12 fourth formers, 10 fifth formers, 4 sixth formers and 2 seventh formers in the sample.

c. A quick and often effective method of sampling is the **systematic** method where two numbers are selected randomly, as the following example shows:

Example E: To select a sample of three from the following list of 20 names:

Anne	Arthur	Ben	Bill
Bob	Bonny	Chris	Den
Evan	Felicity	Frank	George
Hilda	Ian	Jill	John
Kirk	Lenore	Paul	Tania

Select two random numbers, each less than 20.
Suppose random number tables give the values 16 and 7. Counting in rows the 16th name is John.
Counting another 7 names gives Ben (after Tania start at the top of the list again). Counting another 7 from Ben gives Felicity as the third name. Cross out names chosen as you proceed.

d. Often a population will be distributed in subsets, each of which has similar characteristics. In this case such a subset is often randomly chosen and then the survey is conducted on this subset using one of the other methods discussed. Such subsets are called **clusters**.

Example F: A country has three large cities with similar populations. A survey has to be taken. Suggest how the cluster method could help.

Solution: One of the cities is randomly chosen then the sample is taken from that city. The results obtained will be just as accurate as a survey which incorporated the citizens of all three cities and will be easier to organise.

Exercise 33

1. a. A set of values is said to be *randomly* obtained from a population. What does this mean?

   b. A researcher decides to find an estimate of the mean weight of pupils in a school by taking a random sample of 50 pupils and finding the mean weight of the sample. Which of the following methods of sampling ensure a random selection of pupils and which do not? Explain your answer in each case.
      i. Each pupil's name is put on a ticket. The tickets are put in a drum which is well stirred and then 50 tickets are withdrawn.
      ii. Every 10th pupil who walks through the school's main gate is weighed until a sample of 50 has been obtained.
      iii. The weights of the first 50 pupils whose names appear in the detention book are taken.
      iv. The weights of the first 50 pupils who weigh-in for school sports teams are taken.

2. a. A coin is said to be "biased". What does this mean?
   b. How could you test whether or not a coin was "biased"?

   c. A researcher decided to do a survey on school pupils by interviewing only third formers. Which of the following results might be expected to be biased as a result of this? Comment on each.
      i. An estimate of the mean weight of school pupils.
      ii. An estimate of the mean number of cars owned by the parents of school pupils.
      iii. An estimate of the proportion of left-handed school pupils.
      iv. An estimate of the mean number of brown-eyed pupils in the school.
      v. An estimate of the mean height of school pupils.

3. A sample of 100 people has to be chosen to investigate the consumption of alcohol in society. Analyse the following sampling methods for bias:
   a. People are chosen randomly from the telephone book and phoned. Those who reply are interviewed. This is continued until 100 have been questioned.
   b. 100 people are chosen randomly from the telephone book and phoned. If they don't reply, the number is phoned back until a reply is obtained.
   c. 100 people entering a liquor store are randomly chosen and questioned.
   d. 100 people are randomly chosen as they leave a cinema.
   e. 100 people are chosen randomly from the school roll.
   f. Questionnaires are put in letter boxes of a randomly chosen suburb. They are mailed back by people who fill them in. From the questionnaires sent back a sample of 100 is selected.
   g. An advertisement, which includes a questionnaire, is placed in a number of newspapers. People who read it fill it in and send it back, and from the responses a sample of 100 is selected.
   h. 10 streets are randomly chosen from a road map and from the dwellings on these streets, 100 are randomly selected. The houses are visited repeatedly until contact is made with the inhabitants, and then the next person over the age of 15 to have a birthday is questioned.
   i. A list of dwellings in the city is obtained. These are divided randomly into groups of 1 000 and one group is randomly chosen. This group is then divided into groups of 100 and one group is randomly chosen. Each dwelling in this group of 100 is visited. The first person to answer the door is interviewed.
   j. The same method of dwelling selection as in (i) but this time the last person to have had a birthday is interviewed.

4. A sample has to be taken of 50 people to find what their attitudes are to the new government education policy. The sample has to be obtained quickly and reasonably cheaply. It needs to be as unbiased as possible. Comment on the following sampling procedures:
   a. 50 people are chosen from the electoral roll and visited at their homes.
   b. A suburb is randomly chosen and 50 people are selected from the electoral roll in that suburb.
   c. 50 people who work in a large factory are randomly selected from the payroll of the company.

d. A long street is chosen and in every 5th house the next person over the age of 15 to have a birthday is interviewed until 50 people have been interviewed.
e. A large school is randomly chosen, then 50 teachers are systematically selected.

5. A country is inhabited by people of three ethnic groups, A, B and C. The population breakdown is shown below:

Ethnic Group	A	B	C	Total
Number	2 500 000	1 300 000	4 200 000	8 000 000

a. If a stratified sample of 2 000 is to be selected on the basis of ethnic group, how many people from each group should be in the sample?
b. If a stratified sample is chosen which includes 75 members of group A, how many members of the other groups would it contain?
c. A stratified sample is selected so that the number of people from groups A and B total 95. What is the size of the sample?

The following sets of random numbers and names should be used to do exercises 6 to 15.

Random Numbers

7833	3001	1594	7473	1710
9538	0523	6307	9176	8292
0024	6783	1955	9326	3058
0014	6851	3386	5222	1652
1669	4851	6374	4486	9117
4595	0048	7118	9907	4292
5886	3740	1271	5403	0290
5272	8211	7433	8207	7842
7516	5136	8674	0850	9366
3364	0482	9705	7876	7019

Names

Adam	01	Frank	13	Kirsty	25	Sebastian	38
Amanda	02	Gareth	14	Lynette	26	Susan	39
Aroha	03	Gwyneth	15	Luke	27	Tammy	40
Ben	04	Harry	16	Megan	28	Tom	41
Bronwyn	05	Helen	17	Mark	29	Tracey	42
Charlotte	06	Hilda	18	Natasha	30	Valerie	43
Conan	07	Holly	19	Nicholas	31	Verity	44
Danielle	08	Ingrid	20	Odette	32	Warwick	45
Darius	09	Jasmin	21	Oliver	33	Winsome	46
Elaine	10	Jean	22	Paul	34	Winston	47
Edward	11	Jessica	23	Philipa	35	Xenia	48
Fleur	12	John	24	Ramesh	36	Yvette	49
				Rosalie	37	Zane	50

6. Starting from the first number on the first column and going down the columns using the last two digits, find a random sample of five names.

7. Starting from the fifth number of the third column and going down the columns using the first and last digits of the random numbers, find a random sample of four names.

8. If a sample of twenty is required based on the strata of "males" and "females", how many of each type will be required?

9. A sample chosen using the strata "males" and "females" has the following males: {Ben, Harry, Paul, Winston}. How many females must it have?

10. A stratified sample using the strata of "males" and "females" has four males and six females. The males are chosen systematically from a starting point of Edward and recording every seventh name. The females are chosen randomly starting from the third number from the top of the second column and going down the columns using the last two digits of each number. Write down the sample obtained.

11. A stratified sample of fifteen is chosen. The males are chosen randomly starting from the sixth number from the top of the first column, using the second and last digit of each number. The females are chosen randomly starting from the third number of the 2nd column again, going down the columns and using the second and last digits of each number. Write down the sample obtained.

12. A sample of ten is chosen from the cluster of all those with names beginning with letters from A to N. Find the sample if it is chosen systematically, using the seventh such name and every fourth subsequent name.

13. A sample of five is chosen randomly from the cluster of females with names beginning with letters from F to W. Find this sample if the random numbers are selected starting from the eighth number from the bottom of the third column, going down the columns using the middle two digits.

14. Name clusters that the following samples could have been chosen from:
a. {Conan, Gareth, Harry} b. {Odette, Rosalie, Yvette}
c. {Adam, Jean, John, Zane}

15. Describe how this sample could have been chosen systematically: Danielle, Edward, Gareth, Helen and Ingrid.

Problems and Investigations

1. Obtain a census result such as the height of all F3 students at your school.
2. Take samples of about 5%-10% of your total population using different sampling strategies and see how closely the results for each sample **compare** with those of your population.

34. BASIC DATA DISPLAY

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 STATISTICS

- *to collect data and present it visually*
- *to analyse and discuss statistically-based inferences about population or experiments*

Introduction

Once data from a survey have been collected they are often *displayed* so that **deductions** can be made from it. The following example shows **raw data** and is in the typical form of a **tally chart** combined with a **frequency table**.

Example A: In a survey on the attitudes of 25 maths pupils, a student obtained the following data which ranked maths by preference:

Ranking	Tallies	Frequency
1					3
2	卌		6
3	卌 ||	7
4	卌	5
5					3
6			1

Tallies are vertical lines put beside a value each time that value occurs. When four lines have been put down the fifth is usually drawn as a horizontal or diagonal line through the previous four. This gives blocks of five tallies which are easy to count.

The **frequency** of any possible value is the number of times it occurs in a survey. It can be found by counting the number of tallies.

Discrete data means that there are no other values possible between the values obtained in the data.

Example B: In example A, the data obtained is discrete because the only possible values are 1, 2, 3, 4, 5 and 6. There are no other values possible between 1 and 2 such as 1.2, or between 2 and 3, etc.

Display of Discrete, Ungrouped Data

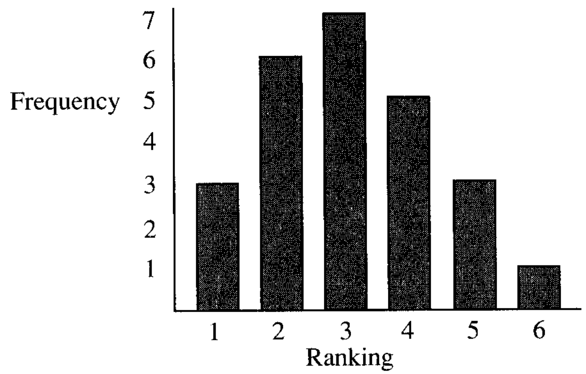
Bar Graphs are the most basic and useful visual display of discrete data.

Bar graphs are drawn using the following general rules:

a. Frequency is usually on the vertical axis.
b. The bars are usually separated when dealing with discrete data.
c. The axes must be labelled and, where appropriate, a title should be given to the bar graph.

Example C: On a bar graph, the data in Example A appears:

Rankings of Maths by Preference



Note: There is a bar for each possible value that the **ungrouped** data has.

Exercise 34a

1. Draw a bar graph of the information given by the following tally chart:

Number of Jobs Held	Tallies
0				
1	卌 卌 卌 卌 ||
2	卌 卌 卌 卌
3	卌 |||
4	卌
5			
6		

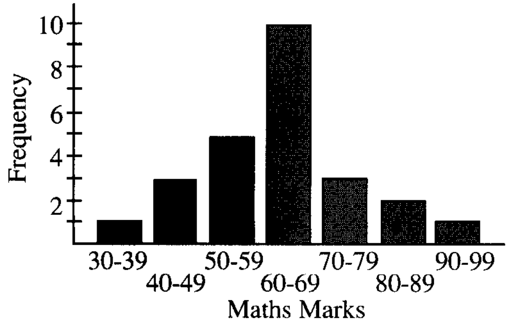
2. Gymnasts in a competition scored the following marks out of 10:
6, 5, 4, 5, 7, 3, 6, 5, 5, 7, 6, 5, 4, 7, 5, 3, 5, 8, 5, 3, 6, 7.
Draw a bar graph to show these results.

Display of Discrete Grouped Data

Grouped data are displayed in groups to show a meaningful trend.

Example D: A student collected the following data about a mathematics test. Rather than recording each mark separately, the student grouped the marks in intervals of 10. A bar graph representing the data is also shown.

Marks	Tallies	Frequency
30-39	I	1
40-49	III	3
50-59	卌	5
60-69	卌 卌	10
70-79	III	3
80-89	II	2
90-99	I	1



Note: The major trends are the concentration of most marks between 50 and 80 and the symmetry of the distribution of marks.

Exercise 34b

1. a. Set up a tally chart as below for the set of numbers and complete it.

Number	Tallies	Frequency
20	I	1
21	III	3
.		
.		
46	I	1

Numbers

27	21	40	37	30	36	24	31	43
36	34	26	40	33	31	43	37	27
34	29	26	46	42	38	40	36	35
34	32	43	23	24	30	40	33	25
21	25	34	39	28	36	40	34	28
21	29	35	31	45	25	43	20	41

b. Use the tally chart you have set up to draw a bar graph.
c. Set up another tally chart as below, with the data grouped in 5s.

Numbers	Tallies	Frequency
20 - 24		
25 - 29		
.....		
.....		
45 - 49		

i. Draw a bar graph for this tally chart.
ii. Comment on what trends or patterns are revealed by each graph.
iii. Which is the most suitable graph and why?

2. The number of workers each day for six weeks at a building site are:

26	27	43	41	53	20	32	27	44	21	29	33
23	23	30	35	54	51	45	37	51	35	36	32
27	24	29	26	46	31	53	26	44	31	37	32
22	24	44	49	30	36						

a. Select a suitable group size for this data and set up a tally chart and frequency table for this data.
b. Draw a bar graph showing this data.

3. The number of absentees daily over the summer term from a 5th form of 97 students at a small high school is listed below:

Number of absentees	Frequency
0 - 4	7
5 - 9	12
10 - 14	16
15 - 19	10
20 - 24	3
25 - 29	6
30 - 34	2
35 - 39	2
40 - 44	5
45 - 49	1

a. Display this data on a bar graph.
b. Determine (if possible) the number of days on which there were:
i. Fewer than 10 students absent.
ii. More than 19 absent.
iii. From 10 - 29 students absent.
iv. Exactly 20 students absent.
v. More than 22 students absent.
vi. More than $\frac{1}{4}$ of the 5th form absent.
c. Explain why it is not always possible to answer questions exactly, such as those in b.

Relative Frequency

The **proportion** of total occurrences of any value is called the **relative frequency** of that value.

Example E: Determine the relative frequencies of the data in Example D and display these in a table.

Solution:

Marks	Relative frequency
30-39	0.04
40-49	0.12
50-59	0.20
60-69	0.40
70-79	0.12
80-89	0.08
90-99	0.04

Note: Total frequency = 25
Frequency of 30-39 is 1
$\therefore$ Relative frequency of 30-39 is $\frac{1}{25} = 0.04$, and so on.

Exercise 34c

1. "Statistics is a branch of mathematics which in some countries is considered to be so important that it is studied as a subject in its own right. The new sixth form syllabus recognises this by devoting far more time to this topic." Analyse this paragraph by setting up a tally chart as below in order to determine the relative frequencies of the vowels in the English language.

Vowel	Tallies	Frequency	Relative Frequency
a			
e			
i			
o			
u			

Using your tally chart, draw a suitable bar graph.

2. The table below shows the major causes of death in New Zealand during 1981. Figures are given as deaths per million.

Cause of Death	1981
Heart disease	2699
Cancer	1770
Stroke	925
Accident	455
Pneumonia	290

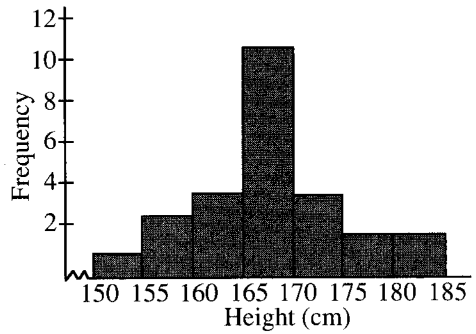
Draw a bar graph showing the relative frequency of deaths per million people during 1981.

Continuous Data

Continuous data means that for any two possible values of data which can be listed, another can always be found between them. Such data are displayed on bar graphs called **histograms**. The bars in histograms *touch* because the data are continuous.

Example F: A student collected data about the heights of pupils in her class. The results are displayed below both in a table and on a histogram.

Height (cm)	Tallies	Frequency
150 < 155	I	1
155 < 160	III	3
160 < 165	IIII	4
165 < 170	~~IIII~~ ~~IIII~~ I	11
170 < 175	IIII	4
175 < 180	II	2
180 < 185	II	2



Note: 150 < 155 means
$150 \leq \text{height} < 155$

Note: The **compression** of the bottom axis. This is a frequently used technique in statistics. It is acceptable only when done to the horizontal axis of bar graphs and histograms but should not be done with vertical axes.

Exercise 34d

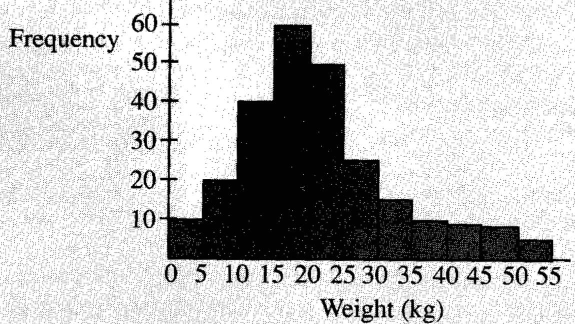
1. Which of the following are *discrete* and which are *continuous*?
   a. Year of birth
   b. Weight
   c. Sex
   d. Exact age
   e. Number in family
   f. Shoe size
   g. Surname
   h. Amount of time spent watching T.V.
   i. Age to nearest minute
   j. Number in families of children in Form 3 classes

2. Measurements are made on the heights of 100 children. The results are given in the table below:

Height (cm)	Frequencies
$160 \leq h < 161$	2
$161 < 162$	3
$162 < 163$	12
$163 < 164$	28
$164 < 165$	26
$165 < 166$	12
$166 < 167$	9
$167 < 168$	5
$168 < 169$	3

Draw a histogram showing the distribution of heights.

3. This histogram shows the data obtained by weighing a large group of children under the age of 14.



   a. How many children weighed between 15 and 20 kg?
   b. How many weighed less than 30 kg?
   c. What number weighed between 10 and 40 kg?
   d. What fraction of the children weighed more than 15 kg?
   e. What fraction of the children weighed between 10 and 30 kg?
   f. What fraction of the children weighed more than 17.5 kg?
   g. Estimate what fraction weighed between 11.25 and 33.75 kg.
   h. Estimate the proportion that weighed between 18 and 30.5 kg.

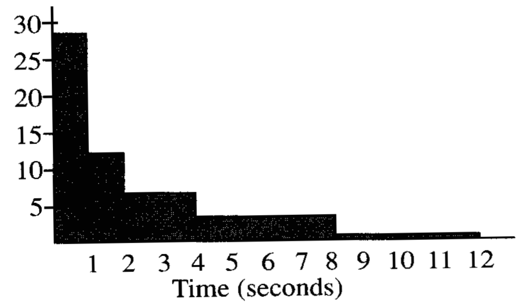
   i. Estimate what percentage of the children weighed between 7.3 and 22.6 kg.
   j. Estimate what percentage of the children weighed between 26.2 and 47.8 kg.

Non-Equal Intervals

Some data have groups or intervals which are not equal. When these data are displayed, *the area of the bars is made proportional to frequency.*

Example G: A survey was taken of the times between successive vehicles passing under a bridge. The following chart shows the distribution of these times, and a histogram shows the results.

Time (s)	Tallies	Frequency
$0 < 1$	卌 卌 卌 卌 卌 III	28
$1 < 2$	卌 卌 II	12
$2 < 4$	卌 卌 II	12
$4 < 8$	卌 卌 II	12
≥ 8	III	3



Note: a. $0 < 1$ means $0 \leq$ time < 1 etc.
b. The height of the bar above the interval 2 to 4 is *half* that of the bar above 1 to 2 because the interval is *twice* as long. Likewise, the height of the bar above the interval 4 to 8 is one quarter that of the interval 1 to 2 because the interval is four times as long.
c. A maximum value (12 seconds) is chosen for time because one is not given.

Exercise 34e

1. A fisherman weighed a sample of his catch and obtained the following data:

Weight (kg)	Frequency
$0 < 5$	35
$5 < 10$	22
$10 < 15$	16
$15 < 20$	12
$20 < 30$	8
$30 < 50$	4
$50 < 100$	18

Note: $0 < 5$ means $0 \leq W < 5$, etc.

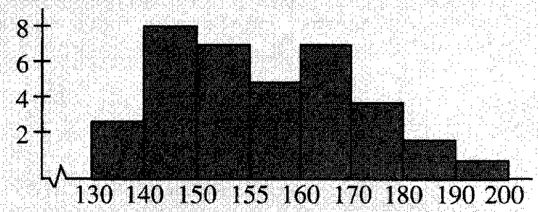
Draw a histogram which accurately shows the distribution of fish in his catch.

2. A boy in 6C took a survey of the heights of his classmates. His data was shown in the following chart:

Height (cm)	Frequency
130 < 140	4
140 < 150	8
150 < 155	7
155 < 160	5
160 < 170	7
170 < 180	4
180 < 190	2
190 ≤ 200	1

Note: 130 < 140 means 130 ≤ height < 140, etc.

The histogram he drew to show this data is drawn below.



a. Write down all the things that are wrong with it.
b. Redraw it so that it more accurately represents the distribution of heights in the class.

3. **Project Work**
If you have done any questionnaires as suggested in the previous chapter, then you should present some of the data obtained with the use of tally and frequency tables, bar graphs and histograms.

35. CENTRAL TENDENCY AND SPREAD

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 7 STATISTICS

- *to calculate sample statistics, including mean and standard deviation, and verify these by reference to a data distribution*

Introduction

Two important concepts used to analyse statistical data are considered in this chapter. They are:

Central tendency which measures the 'central value' of a **sample** or **population**.
Spread or **dispersion** which measures how the data are spread out from the centre in a sample or population.

The central value tells nothing about how the values are spread around that centre as the following example shows:

Example A: Bar graphs have been drawn for each of the following sets of values. x represents the possible values and f represents the frequency with which each possible value occurs.

a.

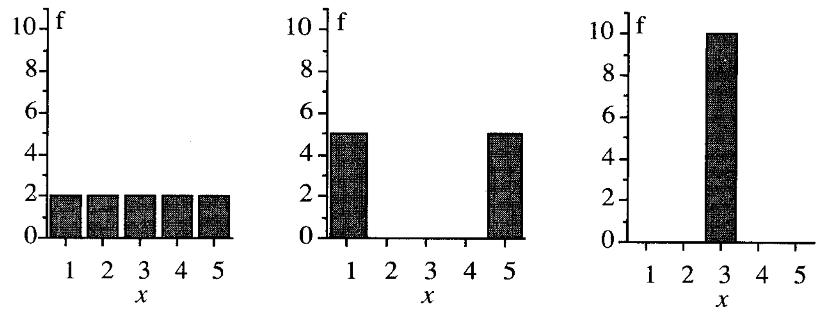
x	1	2	3	4	5
f	2	2	2	2	2

b.

x	1	2	3	4	5
f	5	0	0	0	5

c.

x	1	2	3	4	5
f	0	0	10	0	0



a. Central value is 3. Values are *evenly* spread around the centre

b. Central value is 3. Values are the *most* spread because more values are further from the centre than (a) or (c).

c. Central value is 3. Values are the *least* spread because they are all at the centre, 3.

Measures of Central Tendency

The mean

The **mean** is the sum of all the values divided by the total number of values.

The mean of a *sample* is given the symbol $\bar{x}$ and is often called the **average.** In simple cases, where $x_1, x_2, ..., x_n$ are the different values in the sample and n is the number in the sample:

$$\bar{x} = \frac{\sum x}{n} = \frac{x_1 + x_2 + ... + x_n}{n}$$

Example B: For the sample 1, 2, 7, 12, 13 the mean is $\bar{x} = \frac{1+2+7+12+13}{5} = 7$.

Usually, each value is repeated more than once in which case the following formula is more appropriate:

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

The mean of a whole *population* has the symbol μ (pronounced "mew") and is found using the same method as above.

Example C: The following results are a sample of marks from a 3rd form History test. x stands for the mark scored out of 10 and f for the frequency.

x	0	1	2	3	4	5	6	7	8	9	10
f	1	1	2	5	12	18	16	10	5	4	1

The mean mark is:

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \quad \text{[where } f_i \text{ is the frequency of the ith } x \text{ value, } x_i]$$

$$= \frac{1\times0 + 1\times1 + 2\times2 + 5\times3 + 12\times4 + 18\times5 + 16\times6 + 10\times7 + 5\times8 + 4\times9 + 1\times10}{1+1+2+5+12+18+16+10+5+4+1}$$

$$= \frac{410}{75}$$

$$= 5.47$$

The mean is the most commonly used measure of central tendency. It fits very well into most applications of statistics.

Exercise 35a

1. Find the means of the following sets of numbers:
   a. -5, 12, 13, 5, 6, 11, 12, 27   b. 49, 23, 47, 68, 113, 226, 14
   c.

number	3	4	5	6
frequency	13	22	47	25

2. A student needs to have a mean mark of 58% to get into a course. The student is sitting six subjects and on the first five has got marks of 47, 51, 53, 49, 68. What mark will need to be obtained to get into the course?

3. Write down a set of 5 different numbers whose mean is 23.

Medians, Quartiles and Percentiles

The **median** is the middle value of a population or sample. The median is found by listing the values in numerical order.

Example D: Find the median of the numbers: a. 3, 1, 2 b. 2, 3, 4, 1

Solution:
a. Placing 3, 1, 2 in numerical order gives 1, 2, 3.
   The median is 2 because it is the middle number.
b. In numerical order, the numbers 2, 3, 4 and 1 are 1, 2, 3, 4.
   Because there is an even number of values, the median is found by taking the *mean of the middle pair* of values. Thus the median is $\frac{2+3}{2} = 2.5$.

It can be tedious to list a large sample or population in order. The following example shows an alternative method.

Example E: Find the median of the values in Example C.

Solution: There are 75 values in Example C. The middle number of these 75 values in order is the 38th value. From the frequency table the 38th value must be 5. Hence the median is 5.

The mean is generally easier to find than the median. However, if unusually large or small values occur, the mean can be an untrue indicator of the centre of the sample. In such cases, the median is preferred because it is not affected by these values.

The **quartiles** are two numbers associated with the central tendency of samples and populations.

The **lower quartile** is the value which 25% of the sample or population is **less than** or equal to. This value is the *median* of the lower half of the values.

The **upper quartile** is the value which 75% of the sample or population is less than or equal to. This value is the *median* of the upper half of the values.

Example F: Find the lower and upper quartiles of the set {1, 3, 5, 4, 9, 7, 1, 4}.

Solution:

i. Put the set into order: {1, 1, 3, 4, 4, 5, 7, 9}.
ii. Count the number of elements. In this case there are 8.
ii. Find 25% of the number of elements [25% of 8 = 2].
iv. Find the second number. This is 1. The lower quartile is 1.
v. Find 75% of the number of elements [75% of 8 = 6].
vi. The sixth number in the set is 5. The upper quartile of this set is 5.

If the number of the data values is not divisible by four, the upper and lower quartiles are found by a simple rule of thumb which is illustrated in the following example.

Example G: Find the upper and lower quartiles for the values in Example C.

Solution: Add one to the number in the sample giving 76 [75 + 1 = 76].

Take 25% of 76. This gives 19. The lower quartile is the19th number in numerical order, which is 4.

Take 75% of 76. This gives 57. The upper quartile is the 57th number in numerical order, which is 7.

Percentiles are generalisations of the median and quartiles. Generally the 1st percentile is the value which 1% of the sample or population is less than or equal to; the xth percentile is the value which x% of the sample or population is less than or equal to. Thus:

The lower quartile is the **25th percentile.**
The median is the **50th percentile.**
The upper quartile is the **75th percentile.**

The **mode** is the most frequently occurring value. In Example C, the mode is 5 because it occurs 18 times.

The mode can be a very inaccurate measure of central tendency and hence caution should be applied in using it. It can be used in circumstances where one value is very much more common than all others, such as Example C.

Exercise 35b

1. Find the median, mode, upper and lower quartiles of each of the following sets of numbers.
   a. 23, 32, 13, 16, 73, 42
   b. 4.3, 3.12, 4, 2.7, 4.3
   c.

number	0	1	2	3	4	5
frequency	11	23	42	55	27	8

   d. 64, 4, 34, 13, 78, 13, 50, 62, 73, 42, 14, 45, 69, 50, 85, 63, 56, 88, 0, 77, 91, 32, 31, 85, 34, 45, 12, 47, 41, 70

2. Write down sets of 13 numbers which have:
   a. A median of 46.
   b. A median of 53.
   c. An upper quartile of 79.
   d. A lower quartile of 87.
   e. 31st percentile of 12.

Measures of Spread

The **range** is found by subtracting the smallest value in the sample or population from the largest value.The range is the simplest of the measures of spread though it is not of much use as it rarely tells us how spread out most values are. The following example shows this.

Example H: In example A, the range of a. is 5 – 1 = 4, b. is 5 – 1 = 4 and c. is 3 – 3 = 0
Note: a. and b. have the same range, yet as Example A shows, different spreads.

The **interquartile range** is the difference between the upper and lower quartiles. The interquartile range for the data in Example C follows directly from Example G. The interquartile range is:
upper quartile – lower quartile = 7 – 4 = 3

The **variance** of a sample (symbols s^2 or σ^2) is the mean of the squares of the distance of the data values from the centre. If x refers to the values in the sample, f is the respective frequencies of those values and $\bar{x}$ is the mean of the sample:

$$\text{variance} = \frac{\sum f(x-\bar{x})^2}{\sum f} = \frac{\sum_{i=1}^{n} f_i(x_i-\bar{x})^2}{\sum_{i=1}^{n} f_i}$$

For simple cases the variance can be written; variance = $\frac{\sum(x-\bar{x})^2}{n}$, or it can be written in full, as above, in more complex cases. With practice, the variance can be easily found from the formula. In the following example the variance calculation is done directly by first finding the mean then using the variance formula. In a second method, a table is used to calculate the variance.

Example I: The following table is a sample from a much larger population.

x	1	2	3	4	5
f	4	3	2	3	4

Find the variance of the sample values.

Direct Solution:

$$\text{Mean} = \frac{4\times 1 + 3\times 2 + 2\times 3 + 3\times 4 + 4\times 5}{4+3+2+3+4} \qquad \text{[substituting in } \bar{x} = \frac{\sum fx}{\sum f}\text{]}$$
$$= \frac{48}{16}$$
$$= 3$$

$$\text{Variance} = \frac{\sum f(x-\bar{x})^2}{\sum f}$$
$$= \frac{4(1-3)^2 + 3(2-3)^2 + 2(3-3)^2 + 3(4-3)^2 + 4(5-3)^2}{4+3+2+3+4}$$
$$= \frac{4(-2)^2 + 3(-1)^2 + 2(0)^2 + 3(1)^2 + 4(2)^2}{16}$$
$$= \frac{16+3+0+3+16}{16}$$
$$= 2.375$$

Table Solution:

	x	f	f.x	$(x-\bar{x})$	$(x-\bar{x})^2$	f.$(x-\bar{x})^2$
	1	4	4	-2	4	16
	2	3	6	-1	1	3
	3	2	6	0	0	0
	4	3	12	1	1	3
	5	4	20	2	4	16
Totals		16	48			38

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{48}{16} = 3 \qquad s^2 = \frac{\sum f(x-\bar{x})^2}{\sum f} = \frac{38}{16} = 2.375$$

Note: The calculation of $\bar{x}$ must be done before the table can be completed.

The **standard deviation** (symbol s) is the square root of the variance.

In Example I, the standard deviation is:

$s = \sqrt{2.375}$

$s = 1.541$ (3 d.p.)

Use of Calculators

The standard deviation is the most important measure of spread. Calculation of the standard deviation follows directly from calculation of the variance. It is easily calculated with a calculator which has a statistical mode. The following sequence of key strikes are with a Casio *fx*-82. Owners of other calculators should consult their owners manuals.

MODE . [this puts the calculator into 'statistical mode']

SHIFT SAC [this sequence clears the memory]

1 × 4 M+ [this puts four 1s into the memory]

2 × 3 M+ [this puts three 2s into the memory]

[Continue this process until all data is entered]

$\bar{x}$ gives the mean and σ_n gives the standard deviation.

When dealing with an entire population, the symbol used for the standard deviation is the Greek letter σ, pronounced 'sigma'.

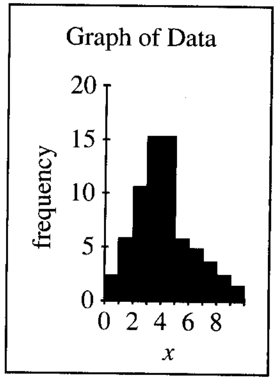
Most calculators have a choice of two buttons for calculating standard deviations. In most situations it does not matter which button is used. If the number of values in a sample is larger than about 20 the difference is usually insignificant.

Use of Spreadsheets

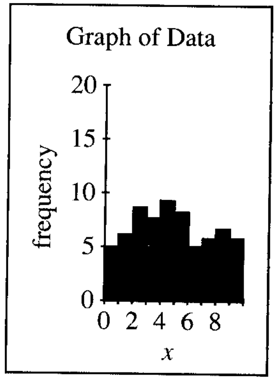
Spreadsheets provide a very effective means of showing the effects of changes in data on standard deviation and bar graphs. In example J the author was able to change the frequencies at will and watch the standard deviation automatically change, as well as the bar graph.

Example J: Demonstration of the Effect of Changes in Frequency on Bar Graph and Standard Deviation using Spreadsheets.

Value (x)	Freq. (f)	fx	x − m	f(x − m)²
0	2	0	-3.58	25.66
1	6	6	-2.58	40
2	11	22	-1.58	27.53
3	16	48	-0.58	5.421
4	16	64	0.418	2.794
5	6	30	1.418	12.06
6	4	24	2.418	23.39
7	3	21	3.418	35.05
8	2	16	4.418	39.04
9	1	9	5.418	29.35
Totals	67	240		240.3
	Mean (m)	3.6		3.587
			St. Dev	1.894



Value (x)	Freq. (f)	fx	x − m	f(x − m)²
0	5	0	-3.58	64.16
1	6	6	-2.58	40
2	8	16	-1.58	20.02
3	7	21	-0.58	2.372
4	9	36	0.418	1.572
5	8	40	1.418	16.08
6	5	30	2.418	29.23
7	6	42	3.418	70.09
8	7	56	4.418	136.6
9	6	54	5.418	176.1
Totals	67	301		556.3
	Mean (m)	4.5		8.303
			St. Dev	2.881



Note how the data in the first diagram is clustered around the centre and has a smaller standard deviation than that in the second diagram which is much more spread out.

Exercise 35c

1. Consider the scores {0, 1, 2, 2, 2, 3, 5, 7, 8, 10}
   a. Find the median, mode and range of the scores.
   b. Copy down and complete the table to show that:
      i. the mean is 4, and
      ii. the standard deviation is 3.16.

x	$(x-\bar{x})$	$(x-\bar{x})^2$
0		
1		
2		
2		
2		
3		
5		
7		
8		
10		
Totals		

2. Consider the scores {2, 3, 3, 4, 4, 4, 6, 7, 8, 9}
   a. Find the median, mode and range of the scores.
   b. Copy down and complete the table to show that:
      i. the mean is 5,
      ii. the standard deviation is 2.24.

x	f	f.x	$(x-\bar{x})$	$(x-\bar{x})^2$	f.$(x-\bar{x})^2$
2					
3					
4					
6					
7					
8					
9					
Totals					

3. a. The mean of {0, 1, 2, 3, 4} is 2. Calculate the standard deviation of the set of numbers without using a calculator.
   b. There are 1 000 marbles in a box; 200 each are marked '0', '1', '2', '3' and '4'. What is the standard deviation of the numbers on the marbles in the box?
   c. Mary took a random sample of 30 marbles from the box and recorded their values in a frequency table:

Number on marble	0	1	2	3	4	total
Frequency	6	5	3	6	10	30

   Calculate the mean of Mary's sample.

d. Without doing any further calculation, say whether the standard deviation of Mary's sample is larger or smaller than that of all the marbles in the box. Explain why.

4. Find sets of 4 digits with standard deviation of:
a. 0 b. 1 c. 2

Adding a Constant Number to Every Value

The effect of adding a constant number to each value is to increase the central value while leaving the spread of values the same.

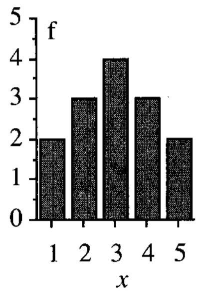
Example K: The following set of values is changed by adding the constant 5 to each possible value while leaving the frequencies unchanged.

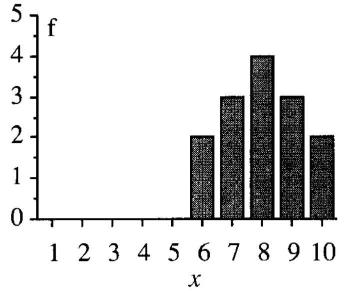
x	1	2	3	4	5
f	2	3	4	3	2

(+5) ⟶

x	6	7	8	9	10
f	2	3	4	3	2

The bar graphs representing each table are:





Note: The centre has increased by 5 and the spread has remained the same.

Multiplying every Value by a Constant

The effect of multiplying every value by a constant is to increase both the centre and the spread of values.

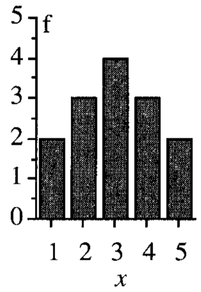
Example L: The original values in Example K are each doubled without changing the frequency.

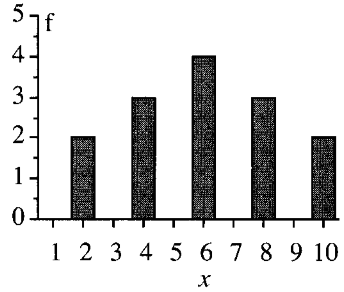
x	1	2	3	4	5
f	2	3	4	3	2

(×2) ⟶

x	2	4	6	8	10
f	2	3	4	3	2

The bar graphs representing each table are:





Note: The centre and the spread have both doubled.

Summarising the results in the previous two examples:

If $x_1, x_2, x_3, ..., x_n$ is a set of values with mean $\bar{x}$ and standard deviation s then:

a. The values obtained by adding the same number k to each are: $x_1 + k$, $x_2 + k$, $x_3 + k$, ..., $x_n + k$ and have a mean $\bar{x} + k$ and standard deviation s.
b. The values obtained by multiplying each by the same number k are: kx_1, kx_2, kx_3, ..., kx_n and have a mean $k\bar{x}$ and standard deviation ks.

Exercise 35d

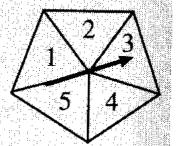
1. a. Find the mean and standard deviation of {1, 2, 5, 5, 7, 10}.
   b. i. Find the new mean and standard deviation if 10 is added to all of the scores, i.e., if the scores are {11, 12, 15, 15, 17, 20}.
      ii. How are the new mean and standard deviation related to the original mean and standard deviation?
   c. i. Find the new mean and standard deviation if the scores are multiplied by 10, i.e., if the scores are {10, 20, 50, 50, 70, 100}.
      ii. How are the new mean and standard deviation related to the original mean and standard deviation?

2. A set of scores has a mean of 6 and a standard deviation of 2. Find the new mean and standard deviation if:
   a. 5 is added to all of the scores.
   b. all of the scores are multiplied by 5.
   c. 20 is added to all of the scores.
   d. all of the scores are multiplied by 20.
   e. 2 is subtracted from all of the scores.
   f. all of the scores are divided by 2.
   g. the frequencies of each score are trebled.

3. a. Find the median, mode, range, mean and standard deviation of {2, 2, 4, 4, 4, 8}.

b. Use your values from (a) to find the corresponding values for:
   i. {5, 5, 7, 7, 7, 11}
   ii. {6, 6, 12, 12, 12, 24}
   iii. {102, 102, 104, 104, 104, 108}
   iv. {200, 200, 400, 400, 400, 800}
   v. {0, 0, 2, 2, 2, 6}
   vi. {1, 1, 2, 2, 2, 4}

4. Mark made a spinner like that shown in the diagram. A pin at the centre of the regular pentagon allows the arrow to be spun around. It soon comes to rest pointing to one of the numbered triangles.



a. i. Copy the table and show what frequencies he expects to obtain after 50 spins.

Number	1	2	3	4	5	total
Frequency						50

ii. Calculate the mean and standard deviation of these numbers.

b. Mark actually obtained these results, with mean 3.04:

Number	1	2	3	4	5	total
Frequency	6	12	14	10	8	50

Is his standard deviation smaller or larger than expected? Explain why, without doing any calculation.

c. Mark's sister changed the numbers to {11, 12, 13, 14, 15}. How would this change:
   i. the mean?
   ii. the standard deviation?

Estimating the Mean and Standard Deviation

Often data is given in such a way that it is impossible to work out the mean and standard deviation exactly. However, a good **estimate** of the mean and standard deviation can be found. The following example illustrates a typical method.

Example M: The table shows the length L of some steel bars.

Length (m)	Frequency
$0 \le L < 1$	23
$1 \le L < 3$	42
$3 \le L < 7$	63
$7 \le L < 12$	22
$12 \le L < 20$	18
$20 \le L < 30$	9
$30 \le L < 50$	5
$50 \le L < 100$	2

Although the exact values of all the data are not known, an approximation can be found by letting all values in any interval take on the value of the *midpoint* of that interval. Thus all values in the interval $0 \le$ length ≤ 1 are assumed to be 0.5; all values in the interval $1 \le L \le 3$ are assumed to be 2, etc. Putting these values into a calculator or using a table as in the previous example gives: $\bar{x} = 8.06$ and $s = 10.75$.

Exercise 35e

1. Find estimates of mean and standard deviation for the following:

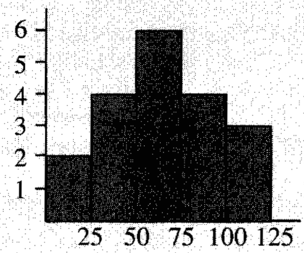
a.

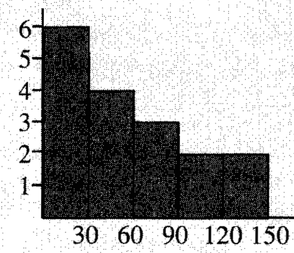
x	0 – 2	2 – 5	5 – 10	10 – 15	15 – 30	30 – 50
f	2	16	25	32	12	3

b.

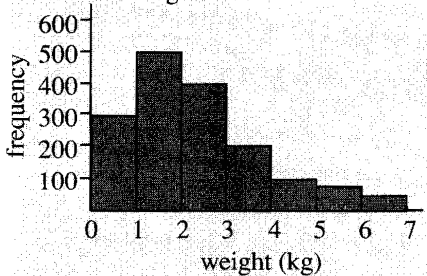
x	-2 – -1	-1 – 1	1 – 5	5 – 7	7 – 9	9 – 12
f	6	3	2	1	2	9

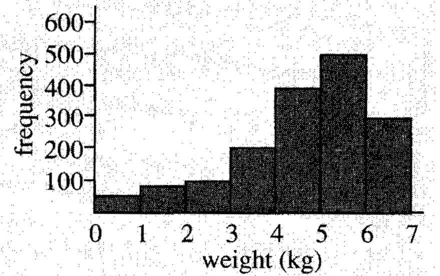
2. Find the approximate mean and standard deviation of the data shown on these histograms:

a.


b.


3. The pair of graphs below shows the distribution of the weights of 2 crops of the same vegetable.





a. Obtain numerical estimates of the mean weight of each crop and also the standard deviations of the weights of each crop.

b. What would the values of the mean and standard deviation of the weights of each crop be if:
   i. the weight of every vegetable had been increased by 1 kg.
   ii. the weight of every vegetable had been doubled.
   iii. there had been twice as many vegetables of every weight in each of the original crops.

Problems and Investigations

1. Get the marks of your class from different tests and compare them using measures of central tendency and spread. Comment on what any differences may indicate.

2. Get the values of something which is published in the newspaper every day such as share prices, world temperatures, etc. and compare them on two different days using measures of central tendency and spread.

36. OTHER DATA DISPLAYS

ACHIEVEMENT OBJECTIVES

On completion of this chapter, students should be able, at:

LEVEL 6 STATISTICS

- *to collect bivariate measurement and discrete number data, and clearly and concisely communicate the significant feaures in appropriate displays, including scatter plots*

LEVEL 7 STATISTICS

- *to collect data, present it visually, and discuss prominent features of the data*

Cumulative Frequency Graphs

Cumulative frequency graphs (sometimes called **ogives**) show the number (or percentage) of a sample or population having any value less than, or less than or equal to, a given value.

Cumulative frequencies are found by *successively adding* the frequencies.

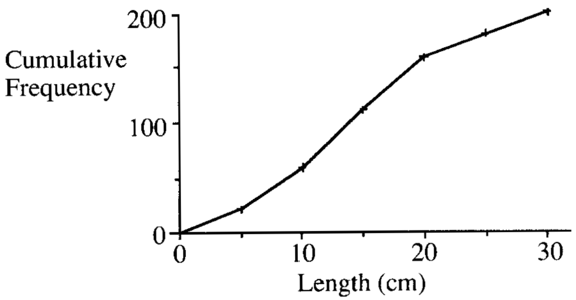
Example A: The distribution of the lengths of 200 fish is shown in the table.

Length (cm)	Frequency	Cumulative frequency
$0 \le L < 5$	22	22
$5 \le L < 10$	38	60
$10 < 15$	52	112
$15 < 20$	48	160
$20 < 25$	21	181
$25 < 30$	19	200

The cumulative frequency 60 is obtained by adding 22 and 38.
The cumulative frequency 112 is 22 + 38 + 52 and so on.

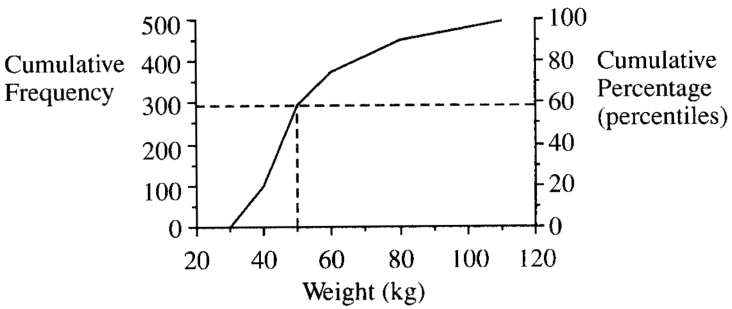
Note: In the above table, $10 < 15$ is a shortened version of $10 \le L < 15$ where L stands for length; similarly $15 < 20$ represents $15 \le L < 20$ etc.

The cumulative frequency graph representing the data in the table is:



Cumulative frequency graphs are particularly useful for finding medians, quartiles and other percentiles.

Example B: The ogive below shows the distribution of the weights of pupils in a school.



The graph shows:

a. The number of pupils weighing 50 kg or less is about 280.
b. About 280 pupils weigh 50 kg or less and about 460 pupils weigh 90 kg or less. The number of pupils weighing *between* 50 kg and 90 kg is thus: 460 – 280 = 180 (approximately).
c. The **median** weight of the pupils at this school corresponds to a cumulative percentage of 50% and is about 47 kg.
d. The **upper quartile** of the weights of the pupils corresponds to a cumulative percentage of 75% and is about 61 kg.

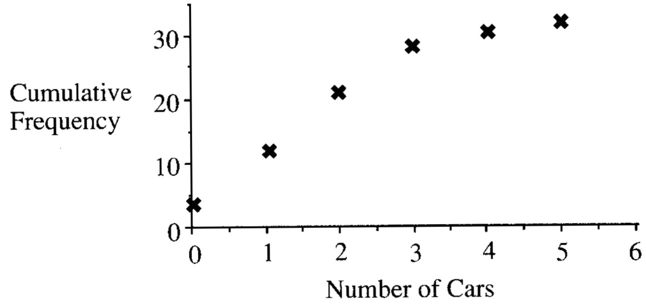
Note: The above ogive showing **continuous** data is a series of successive points joined with straight lines rather than a smooth curve. (It is statistically more correct to use such straight lines).

The cumulative frequency graphs for **discrete** data are drawn as the following example shows:

Example C: Draw a cumulative frequency graph to show the distribution of car ownership in a sample of people as shown by this table of values:

Number of cars owned	0	1	2	3	4	5
Frequency	4	8	10	6	3	1
Cumulative frequency	4	12	22	28	31	32

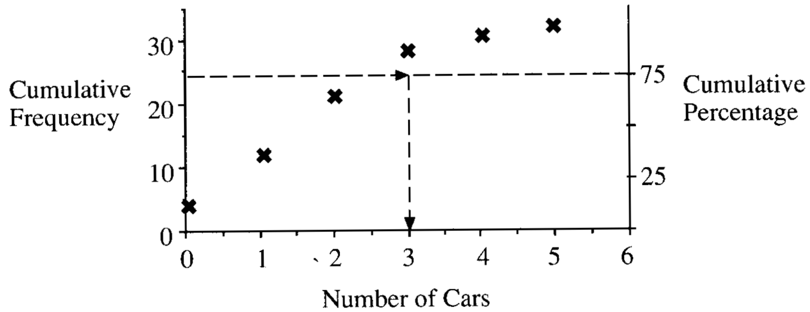
Solution: The cumulative frequency curve is:



At 1 on the 'cars' axis, the cumulative frequency jumps to 12 because the number of people who own 1 or fewer cars is 12. The graph jumps again when 2 cars is reached.

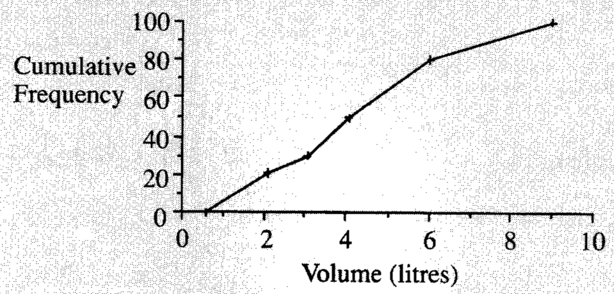
To find any percentile on a cumulative frequency graph, go across from the appropriate position until there is a cross directly above. Going down at this point and reading off the value gives the required result.

Example D: To find the upper quartile (75th percentile) for the data in Example C, begin at 75% of 32 on the vertical axis and move across until below a cross. Going down gives the upper quartile, 3.



Exercise 36a

1.



The above cumulative percentage graph shows the distribution of the amount of oil used by a large fleet of cars in a 6 month period (excluding oil changes).

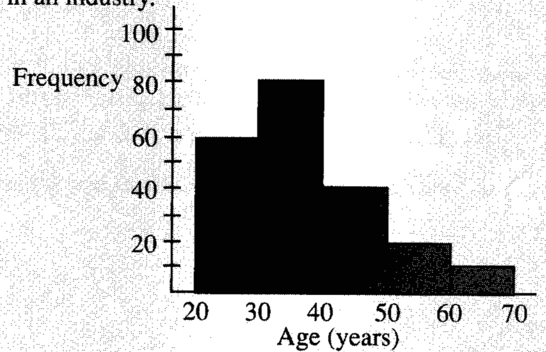
a. What percentage of cars used 4 litres or less?
b. What percentage of cars used 5 litres or more?
c. What percentage of cars used between 3 and 7 litres?
d. What is the median amount of oil used?
e. What is the lower quartile of the amount of oil used?
f. Sketch a histogram showing the distribution of the amount of oil used.

2. The lifespan of a certain species of animal is investigated. The lifespans of 50 such animals which died of natural causes are recorded below:

Lifespan (years)	<1.5	1.5–2.5	2.5–3.5	3.5–4.5	4.5–5.5	5.5–6.5	≥6.5
Frequency	3	4	12	13	11	5	2

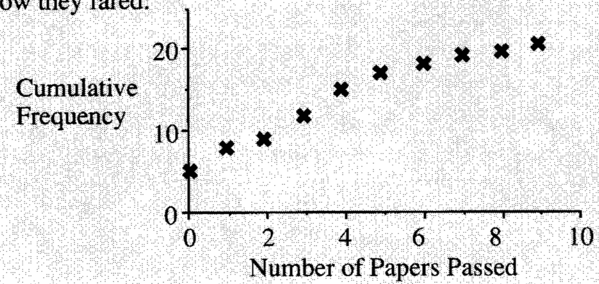
a. Draw a cumulative frequency graph to show the distribution of lifespans.
b. Use your ogive to estimate the median lifespan of these animals.
c. Prior to what age did 70% of these animals die?
d. What percentage of animals died between 2.4 and 5.8 years?
e. What is the interquartile range?

3. The following histogram shows the distribution of ages of qualified technicians in an industry.



a. Draw a cumulative frequency graph to show the same distribution.
b. Estimate the number of technicians older than 32.
c. Estimate the median age.
d. Estimate the interquartile range.
e. Estimate the number of technicians between 25 and 37 years old.

4. A sample of university students was taken to find out how many papers they had passed in their recent exams. The cumulative frequency graph below shows how they fared:



a. What was the number in the sample?
b. How many students passed 6 papers or less?
c. How many students passed between 2 and 8 papers inclusive?
d. What was the median number of papers passed?
e. What was the maximum number of papers that any student passed?

5. A soccer team played 60 games over a period of 3 years. The goals per game they scored are recorded in this table:

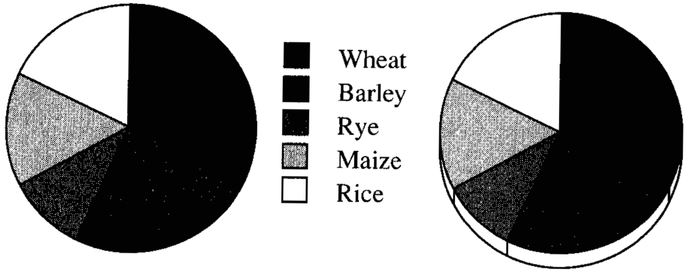
Goals per game	0	1	2	3	4	5	6	7
Frequency	3	5	12	14	10	8	4	4

a. Draw a cumulative frequency graph to show this distribution.
b. Find the median number of goals scored per game.
c. Which is the greater – the median or the mean number of goals scored per game?
d. Find the upper quartile of the number of goals scored per game.
e. In what percentage of games did the team score 3 goals or less?

Pie Graphs

Pie graphs are widely used in many fields of study and in the media to show *proportions*.

Example E: The relative proportions of different types of crops planted on a nation's arable land are shown:



2 dimensional 3 dimensional

The pie graph shows that about $\frac{1}{3}$ of the nation's arable land is planted in wheat, about $\frac{1}{4}$ in barley, and the remainder planted in other crops.

Often the actual percentages are included on the pie graph.

Example F: Draw a pie graph showing the proportions of magazine types in a bookstore. They have the following numbers of each brand on sale:

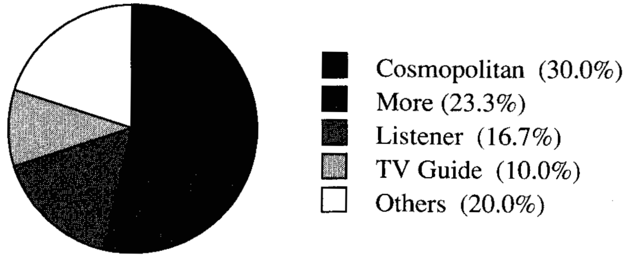
Cosmopolitan	45
More	35
Listener	25
TV Guide	15
Others	30
Total	150

Solution: The sector angles are calculated using proportion.
The sector angle for Cosmopolitan is $\frac{45}{150} \times 360° = 108°$ [360° in a full circle]

Similar calculations lead to the following table of values for sector angles:

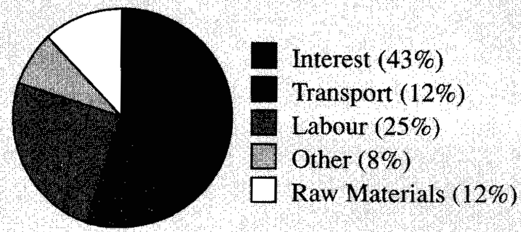
Magazine	Frequency	Percentage	Sector Angle
Cosmopolitan	45	30	108
More	35	23	84
Listener	25	16	60
TV Guide	15	10	36
Others	30	20	72

Using these sector angles the pie graph is completed:



Exercise 36b

1. The pie chart below shows the proportions of a large company's major expenses:



   a. If its total expenses were $63 479 000, find the interest and labour costs.
   b. How large should the sector angle be for transport?
   c. Sketch a bar graph showing this same distribution.
   d. Explain why it is not possible to draw a mathematically valid ogive showing this distribution.

2. A computer sales manager asked her assistant to draw a large pie graph showing the distribution of the different makes of computer they stocked. These are given in the table below:

Make	Apple	IBM	Wang	NEC	Others
Number	65	42	23	12	8

   Draw a pie graph of the type the assistant should have drawn.

3. The following table shows the numbers of people of various ethnic groups living in Ransonville, an inner city suburb of Porkland, largest city of Old Mealand over a 40 year period.

Year	Algols	Fortrans	Logos	Others
1945	6 050	3 830	1 870	250
1965	4 230	2 560	4 890	320
1985	4 623	900	2 320	657

a. Draw pie graphs to show the proportions of different ethnic groups in 1945, 1965 and 1985.
b. State any important trends revealed by your pie graphs.
c. There is one vital piece of information which can be obtained from the table but not from your pie graphs. What is it?

4. The table below shows the numbers of people living in the North and South Islands of New Zealand at the time of various censuses during this century.

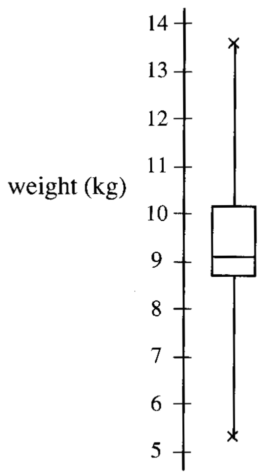
Total Population			
Census Year	North Island	South Island	Total
1901	431 471	384 391	815 862
1945	1 146 315	556 015	1 702 330
1981	2 322 989	852 748	3 175 737

Draw 3 pie graphs showing the percentages of people living in the North and South Islands in:
a. 1901 b. 1945 c. 1981
What do these graphs illustrate?

Box Plots

Box plots are a one-dimensional form of data display useful for getting an idea of the distribution of data. They are also called box and whisker plots.

Example G: Draw a box plot of data whose maximum value is 13.6 kg, minimum value is 5.3 kg, upper quartile is 10.2 kg, lower quartile is 8.7 kg and median is 9.1 kg.

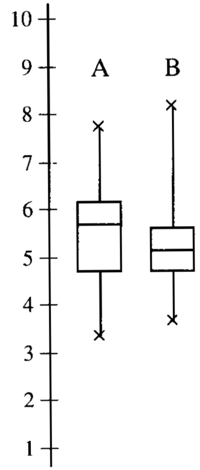


The rectangle has its upper side at the level of the **upper quartile**, the lower side at the level of the **lower quartile**. A further parallel line is drawn through the rectangle at the **median**. This part of the diagram is the 'box'. The 'whiskers' are the two lines which emanate from the box to the maximum and minimum values.

This box plot shows data which is slightly 'skewed' towards the upper end. This means that data above the median is more spread out than the data below it.

Box plots of different sets of data when placed beside each other provide excellent means of comparing them.

Example H:



This diagram shows:
a. A and B to have the same range.
b. A has a greater interquartile range than B.
c. B has a lower median than A.
d. B has maximum and minimum values higher than those of A.
e. A is 'skewed' toward the lower values while B is symmetrical. The term 'sheared' refers to a lack of symmetry between the upper quartile and lower quartile, with regard to the median.

Exercise 36c

1. Draw a box plot of this data:
13, 41, 15, 35, 38, 24, 29, 35, 31, 38, 31, 49, 55

2. a. Draw box plots to compare these marks obtained by 2 different groups of pupils sitting the same exam:
Group A - 90, 12, 28, 5, 5, 74, 93, 1, 53, 18, 4, 89, 49, 29, 68, 87, 47, 92, 24, 84, 6, 2, 2, 16, 0, 61, 25
Group B - 3, 50, 76, 86, 84, 0, 73, 28, 15, 40, 10, 18, 5, 86, 53, 74, 9, 87, 75, 67, 87, 40, 81, 3, 99, 61, 88
b. Comment on any differences shown in the plots.

Stem and Leaf Diagrams

These are similar to tally charts, but provide more information.

Example I: Draw a stem and leaf diagram of the following set of numbers.
{13, 25, 12, 34, 47, 53, 17, 12, 10, 48, 36, 37, 27, 29, 14, 9, 2, 24, 42, 42, 39, 33, 45, 50, 38}

Stems are to be tens and leaves are to be ones.

Solution:

Diagram A (Final version)

```
5 | 0 3
4 | 2 2 5 7 8
3 | 3 4 6 7 8 9
2 | 4 5 7 9
1 | 0 2 2 3 4 7
0 | 2 9
```

Diagram B (Rough version)

```
5 | 3 0
4 | 7 8 2 2 5
3 | 4 6 7 9 3 8
2 | 5 7 9 4
1 | 3 2 7 2 0 4
0 | 9 2
```

Diagram A is largely self-explanatory. Numbers from 10 to 19 are placed beside the 1 in order.

1 | 0 2 2 3 4 7 represents those numbers 10, 12, 12, 13, 14, 17 from the given set.

2 | 4 5 7 9 represents the numbers 24, 25, 27, 29, etc.

[If in a hurry, do a rough stem and leaf plot first then order it properly later. See diagram B above.]

There are many possible choices for stems and for leaves. The choice should be made so as to best show the distribution of data.

Example J:

```
5 | 00
4 | 21 23
3 | 55 63 80
2 | 23 27
1 | 11 23 47 48
```

In this diagram, stems are hundreds and leaves are units.

The set of data the diagram represents is:
{111, 123, 147, 148, 223, 227, 355, 363, 380, 421, 423, 500}.

Exercise 36d

1. a. Draw a stem and leaf diagram with stems being units and leaves tenths for this set of data:
   3.1, 3.3, 2.2, 1.4, 0.2, 3.4, 3.4, 4.5, 4.2, 3.7, 5.6, 4.4, 4.2, 3.5, 3.8, 3.6, 2.6, 2.5, 2.5
   b. Using your diagram or otherwise find the median, upper and lower quartiles.
   c. Draw a box plot of the data.

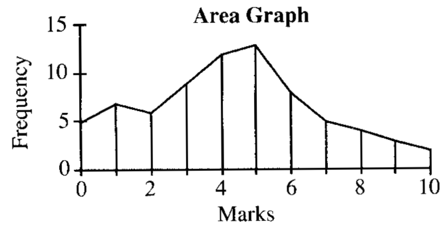
2. The following numbers are the weights in kg of some heavy athletes in the Junior World Championships:
   119, 108, 124, 108, 122, 130, 108, 111, 125, 115, 112, 115, 119, 109, 117, 118, 109, 109
   Draw a stem and leaf diagram for this with stems being tens and leaves units.

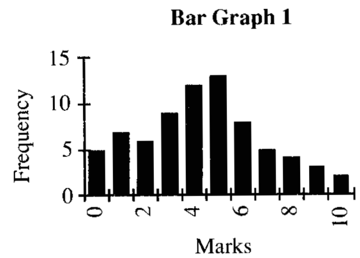
Using Spreadsheets

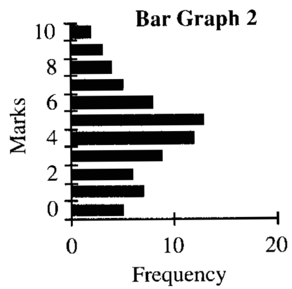
Any student who has access to a spreadsheet when doing work necessitating the display of data should use it. With only a little practice it is possible to produce a great variety of data displays very quickly and neatly. The series of displays on this page were produced with an Excel spreadsheet in the space of ten minutes.

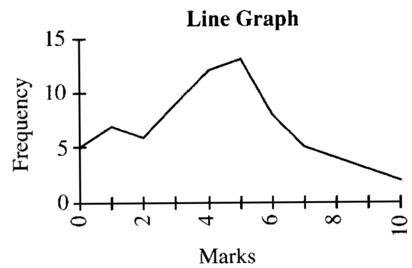
Example K: Demonstration of some of the different types of data display available with spreadsheets.

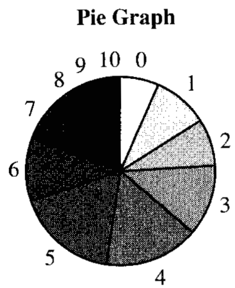
Mark	Frequency
0	5
1	7
2	6
3	9
4	12
5	13
6	8
7	5
8	4
9	3
10	2

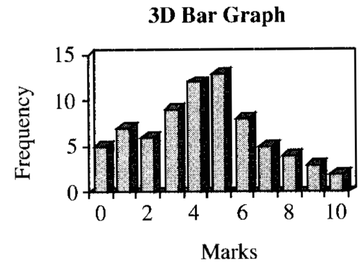




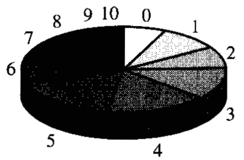




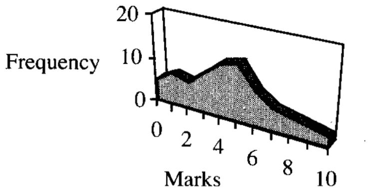




3D Pie Graph



3D Line Graph



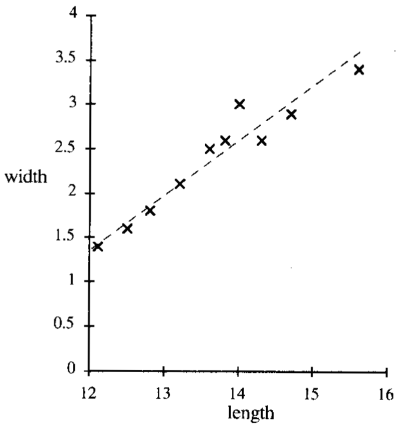
Scatter Diagrams

Scatter diagrams give a quick and quite reliable method for detecting relationships when dealing with paired data.

Example L: The following table gives widths and lengths of leaves from a sample taken from a species of tree.

Length (cm)	12.5	13.2	12.1	14.3	15.6	14.7	12.8	13.6	13.8	14.0
Width (cm)	1.6	2.1	1.4	2.6	3.4	2.9	1.8	2.5	2.6	3.0

The scatter diagram of this data appears as



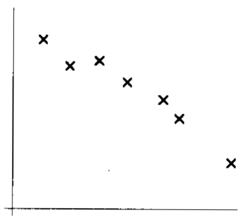
a. Inspecting this graph reveals an obvious trend for width to increase as length increases.

b. The relationship also appears to be linear as all plotted points lie close to the straight line dotted in.

In cases where there is a linear relationship it is possible to find what is called 'the line of best fit' using a technique called *regression*. This topic is outside the syllabus but a reasonable approximation to this line may be achieved. A line is drawn through the data which has as about as many points on one side of it as it does on the other. Common sense must prevail when drawing such a line.

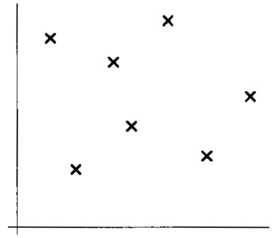
When there is a clear linear relationship with positive gradient we say the two variables are 'correlated positively'. This does not necessarily mean one is influencing the other.

Sometimes we get scatter diagrams which appear like this:



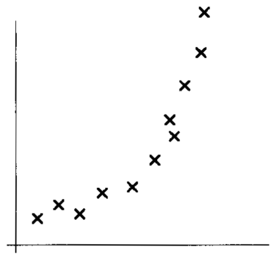
In this case there is a linear trend with negative gradient. The variables are said to be negatively correlated.

Often we get scatter diagrams which have this appearance:



There is no obvious pattern to the distribution of points and we say the variables are uncorrelated.

On occasion the scatter diagram will reveal a trend but one which is clearly not linear.



Exercise 36e

1. Find out if there is any correlation between the number of pages in children's books and their price by drawing a scatter diagram for the following sample:

number of pages	32	32	24	32	16	32	24	16	24	32	24
price	9.50	10.00	7.00	6.00	6.50	4.50	5.50	17.00	15.00	5.00	5.00
number of pages	160	32	32	32	32	32	32	32			
price	17.00	3.50	2.00	3.00	3.50	2.00	3.50	2.50			

2. The following table gives the lifts in the snatch and the clean and jerk for competitors in the 76kg class at a world weightlifting championship. Find out if performance in the two lifts is correlated.

Snatch	110	110	120	115	117	125	122	130	130	135	145
C & J	142	145	140	155	157	160	165	160	157	160	165

Problems and Investigations

Collect samples of paired data such as:

a. height : weight
b. term 1 marks : term 2 marks
c. price in shop A : price in shop B
d. April car price : August car price

Compare the data using box plots and scatter diagrams.

37. MISLEADING USES OF STATISTICS

Achievement Objectives

On completion of this chapter, students should be able, at:

LEVEL 7 STATISTICS

- *to analyse and discuss statistically based inferences about populations and experiments*

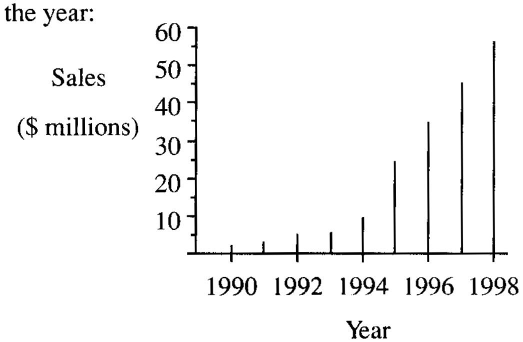
Introduction

It has often been said that a picture is worth a thousand words. This is certainly the case when a trend, pattern or distribution in a table of figures or data is displayed. It is also possible with displays to lead people to conclusions which the data does not justify. Such use of displays may be either accidental or deliberate, and often happens when advertisers or politicians present graphs and other diagrams to 'justify' their claims.

Examples of Misleading Displays

The examples which follow illustrate some situations where it is possible to create an incorrect impression by graphical or diagrammatic means.

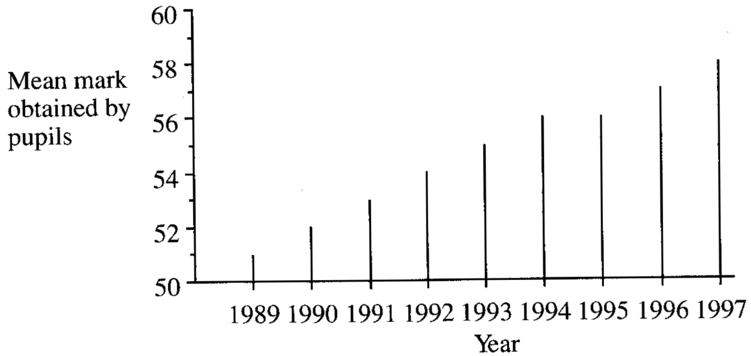
Example A: The following graph shows the sales figures for a company plotted against the year:

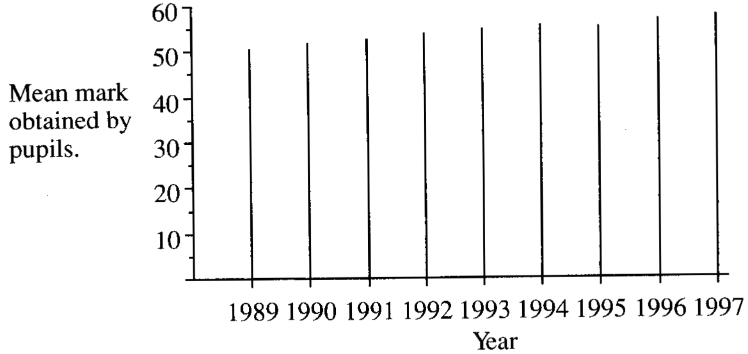


Quite obviously data are not available for sales in years which are still in the future. If the current year is 1994, then this graph could create a false impression of the sales it is supposedly illustrating. This graph is an example of what is called **'extrapolating beyond the data'**, meaning that a trend evident on a graph relating to the past and present is continued into the future, based on the assumption that this trend will continue.

Changing the vertical scale can have a significant effect on how the viewer perceives the data.

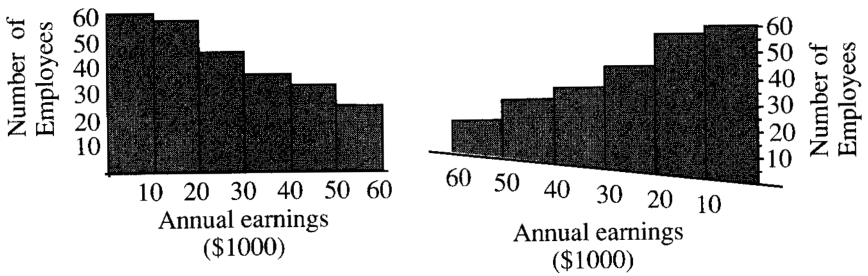
Example B: The same marks obtained in a standard mathematics exam are shown on each of the two bar graphs following:





An *impression* of a spectacular improvement in the exam marks is obtained by starting the vertical scale at 50 rather than 0. This is called a **'suppressed scale'** due to the fact that the vertical axis is plotted over a smaller range of values.

Example C: The two graphs below show the distribution of incomes in a company. They both show exactly the same data but while the left graph is an ordinary histogram, the right one uses slanting scales, the so-called **'advertiser's scale'**.



The slanting scale creates the impression that the number of people with incomes from $0 to $10 000 is about four times that of those with incomes from $50 000 to $60 000. Careful examination of the graphs shows that the number of those earning between $0 and $10 000 is only about twice that of those earning $50 000 to $60 000.

Any type of graph which appears in advertisements, magazines and newspapers should be interpreted carefully. In addition to the factors already mentioned, which can lead to misinterpretation of data display, the following should also be watched for:

a. Absence of scale units on axes.
b. Non-uniform scales.
c. Comparison between graphs with different scales.
d. Lack of titles and labels.
e. Different colour and thickness of bars in bar graphs and histograms.

Another misuse of statistics is the use of two or more sets of data to reach unjustified conclusions, as shown below:

Example D: In the years 1980-1985 the following trends occurred in New Zealand. These can be verified from official statistics.

a. The crime rate rose.
b. The overseas debt rose.
c. The number of young people playing soccer increased.
d. The use of corporal punishment in schools decreased.

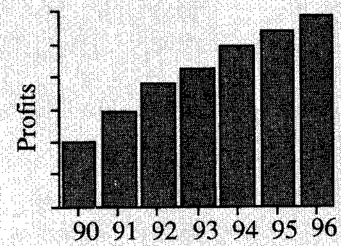
It is incorrect though to make conclusions based on a connection between any pair of these trends without further evidence. Among conclusions which might be reached from the above if things which occurred simultaneously were always connected are:

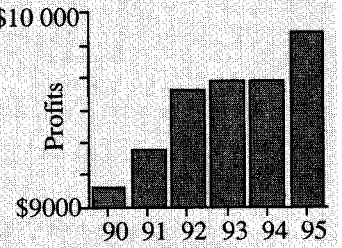
i. The crime rate rose because the use of corporal punishment in schools decreased.
ii. The overseas debt rose because the crime rate rose.
iii. The number of young people playing soccer increased because the overseas debt rose.

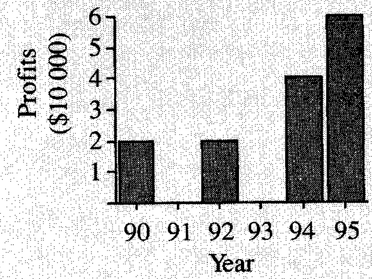
It may well be that each of the above statements is true, but there is no clear evidence to establish whether they are or not. It certainly does not follow that because two things occur at the same time, one is the cause of the other or vice versa. In many cases, both are caused by some other underlying factor.

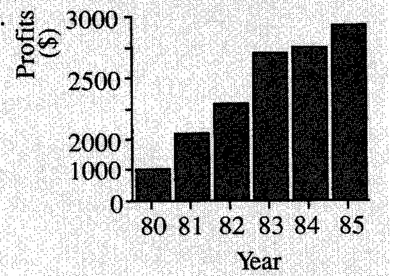
Exercise 37

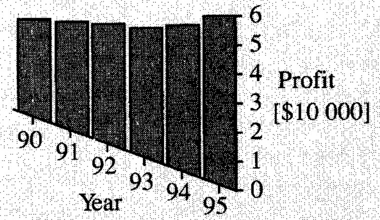
1. Explain how each of these graphs could be misleading. Each purports to show the profit position of Rutherford Enterprises Limited for recent years:

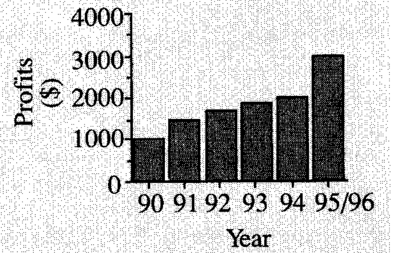
a.


b.


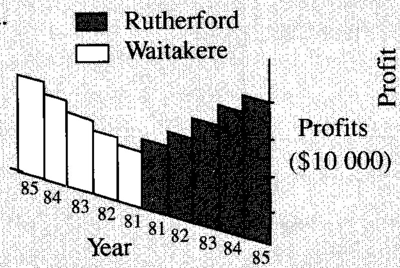
c.


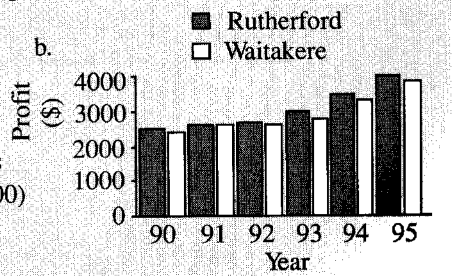
d.


e.


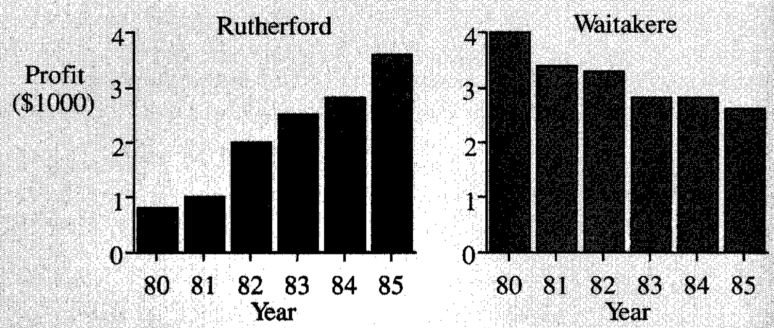
f.


2. These pairs of graphs attempt to compare the profit picture for Rutherford Enterprises against its main competitor, Waitakere Products. Explain how each pair may be or is giving a false picture of the true situation:

a.


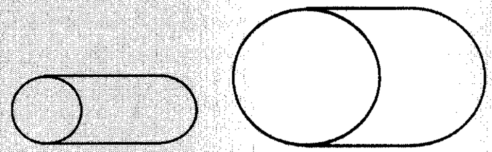
b.


c.



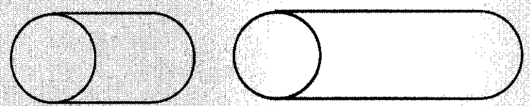
3. In the year 1997 to 1998 Waitakere Products sold half as much oil as Rutherford Enterprises. Which of the diagrams below fairly describe the relative sales picture? For each picture explain why you believe each pair of pictures fairly or unfairly represents what is happening.

a.



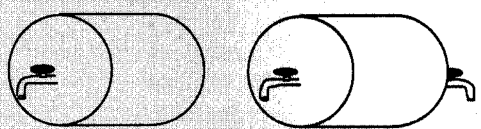
Waitakere Products Oil Sales — Rutherford Enterprises Oil Sales

b.



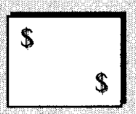
Waitakere Products Oil Sales — Rutherford Enterprises Oil Sales

c.

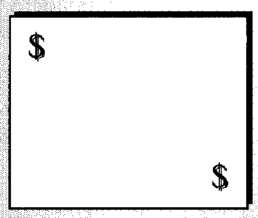


Waitakere Products Oil Sales — Rutherford Enterprises Oil Sales

d.

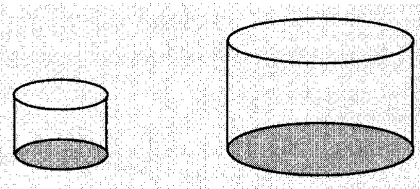


Waitakere Products Oil Sales



Rutherford Enterprises Oil Sales

e.



Waitakere Products Oil Sales — Rutherford Enterprises Oil Sales

Problems and Investigations

Find in magazines and newspapers as many examples as you possibly can of misleading graphs and data displays and comment on why they are misleading.

38. TIME SERIES

Achievement Objectives

On completion of this chapter, students should be able, at:

LEVEL 7 STATISTICS

- *to identify causes of long-term and short-term trends in time series data extracted from reference sources or experiments*
- *to highlight features in time series graphs by simple algebraic transformations*

Time Series

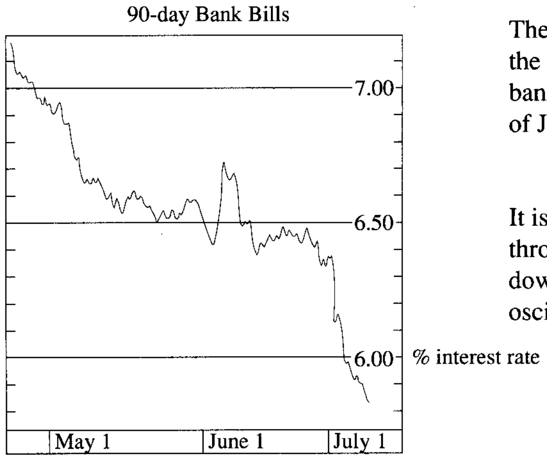
A **time series** is a sequence of values of a variable at different points in time.

Example A:

a. Numbers of cars passing under a motorway bridge during five minutes at the beginning of each hour for a 24 hour period.
b. Number of unemployed at the end of each month for a three year period.
c. Temperature at 12.00pm recorded daily throughout June in a park.

In studying time series it is vital to graph the information to show any patterns more easily.

Example B:



The graph opposite shows the interest rate on 90-day bank bills near the beginning of July 1993.

It is clear that there is, throughout May and June, a downward trend with some oscillation.

Analysis of Time Series

Time series graphs are usually checked for three features:

a. **A secular trend**

This is a general tendency for increase, decrease or stability. In example B, there is a secular downward movement with time.

b. **Periodic movements**

A typical example is oscillation which can result from changes in tides, seasons or times of the day or perhaps the expansion and contraction of economies. In example B there seems to be a minor periodic oscillation of about a day's duration added to the general trend.

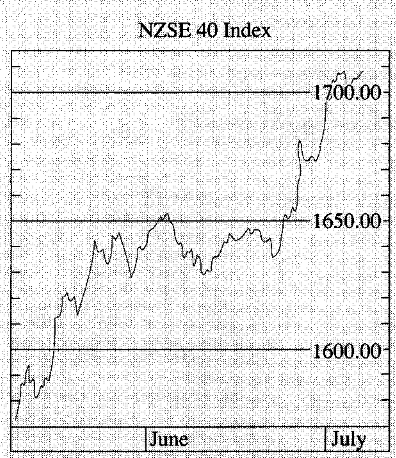
c. **Erratic or residual variation**

These are variations which remain after account has been taken of secular and periodic movements. In example B there seems to be a rebound in early June. The reason for this is not obtainable from the data. There may have been some government or overseas action which reversed the previous trend. An investigation of the newspapers of that time could yield the reason.

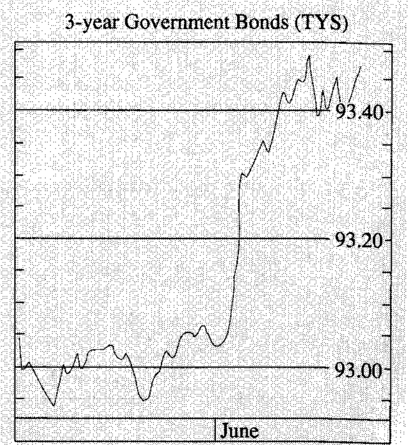
Exercise 38a

Examine the following graphs for trends.

1.



2.



Analysis of Trends

Secular Trend

Data is analysed for secular trend by using what are called '**moving averages**'. By averaging over a number of consecutive periods the effect of variation is greatly minimised. If the number of periods is three then the moving averages are said to be 'of order 3'. If the number of periods is seven then the moving averages are said to be 'of order 7'.

Example C:

a. The following data were obtained. Analyse the data for a secular trend by the method of 'moving averages' over 4 seasons (order 4).

Tonnes	45	37	50	64	53	46	58	68	55
Date	Jan 88	Apr 88	July 88	Oct 88	Jan 89	Apr 89	July 89	Oct 89	Jan 90

b. Predict the value in April 1990.

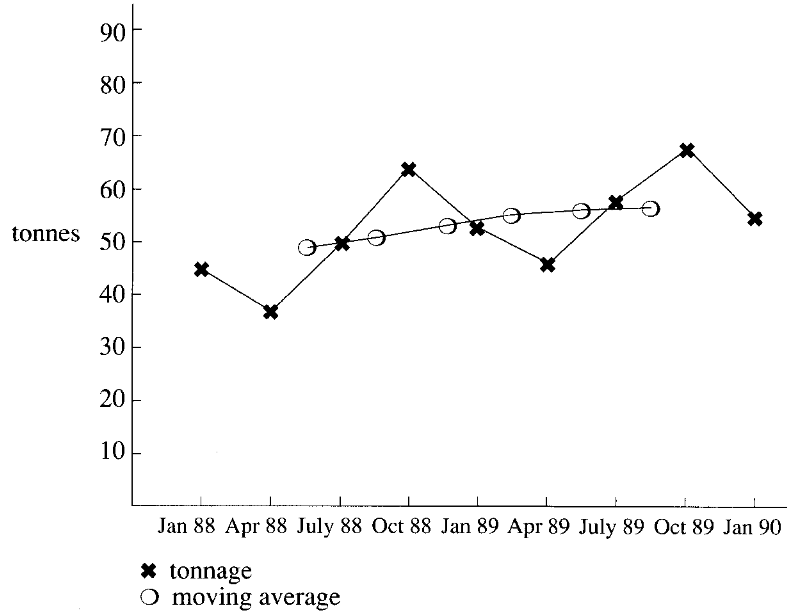
Solution: Averaging over the first four seasons gives a value of $\frac{45 + 37 + 50 + 64}{4} = 49$ which is 'centred' in the middle of the four seasons in May 1988.

Averaging over the next four seasons gives $\frac{37 + 50 + 64 + 53}{4} = 51$ which will be 'centred' in the middle in August 1988.

The following table shows the results of the calculations of all such moving averages.

Tonnes	49	51	53.25	55.25	56.25	56.75
Date	May 88	Aug 88	Nov 88	Feb 89	May 89	Aug 89

A graph using the information from both these tables is now drawn.



There seems to be a linear trend with an increase of $\frac{56.75 - 49}{5} = 1.55$ per three months [we divide by 5 because the moving averages are over 5 time periods].

b. Assuming the linear trend continues we would predict that in November 1989 the moving average will be 56.75 + 1.55 = 58.3.

Let the tonnage in April 1990 be T.

$\therefore \frac{58 + 68 + 55 + T}{4} = 58.3$

$\therefore T = 4 \times 58.3 - 58 - 68 - 55$

$\therefore T = 52.2$

Exercise 38b

1. a. Graph the following time series:

Value	13	15	12	11	16	19	19	13
Time	1	2	3	4	5	6	7	8

b. Calculate the moving averages of order 3.
c. Graph the moving averages on the same graph as the values.

2. a. Graph the following time series:

volume (litres)	25	23	19	14	13	17	16	11	8	6	4	2
time (months)	1	2	3	4	5	6	7	8	9	10	11	12

b. Calculate the moving averages of order 6.
c. Graph the moving averages on the same graph as the volume.
d. Predict a value for the volume in month 13.

Analysis of Periodic Variation

Periodic variation is regular fluctuation or variation about a general trend. A good natural example would be the surface of the water on an incoming tide. The water is rising but on the surface of the water there is regular variation due to waves.

Investigation is usually carried out by a simple technique called '**ratio to moving average**'. Application of this technique to the data from example C will show how it is carried out.

Example D: The data we have is:

Tonnes	45	37	50	64	53	46	58	68	55
Date	Jan 88	Apr 88	July 88	Oct 88	Jan 89	Apr 89	July 89	Oct 89	Jan 90

Moving Average	49	51	53.25	55.25	56.25	56.75
Date	May 88	Aug 88	Nov 88	Feb 89	May 89	Aug 89

In order to calculate 'ratio to moving average' it is first necessary to estimate the tonnes of the moving average dates.

The first date is May 88. This is between April 88 and July 88 so we estimate May's tonnage to be the average of the tonnes for those two months. Similarly with Aug 88, Nov 88, etc.

For May 1988 it is $\frac{1}{2}(37 + 50) = 43.5$. For August 1988 it is $\frac{1}{2}(50 + 64) = 57$

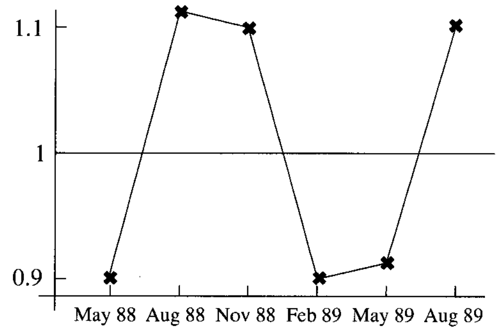
The ratio to moving average for May 1988 is $\frac{43.5}{49} = 0.89$

The ratio to moving average for August 1988 is $\frac{57}{51} = 1.12$

The table below gives the final set of data.

Date	Tonnage	Moving Average	Ratio to Moving Average
May 88	43.5	49	0.89
Aug 88	57	51	1.12
Nov 88	58.5	53.25	1.10
Feb 89	49.5	55.25	0.90
May 89	52	56.25	0.92
Aug 89	63	56.75	1.11

The graph shows ratio against date. The ratio to the moving average is the proportion that the actual value is of the moving average.



This graph, although only covering a fairly short period of time, indicates a periodic component which never seems to deviate from a value close to 10% of the moving average.

Use of Spreadsheets

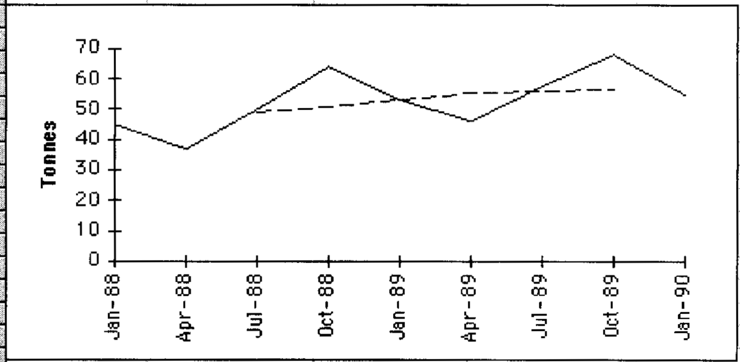
The work in analysing time series for secular trend and periodicity is enormously speeded up by the use of a spreadsheet. The following steps were taken in the creation of the spreadsheet analysis of the data for example C.

Example E:

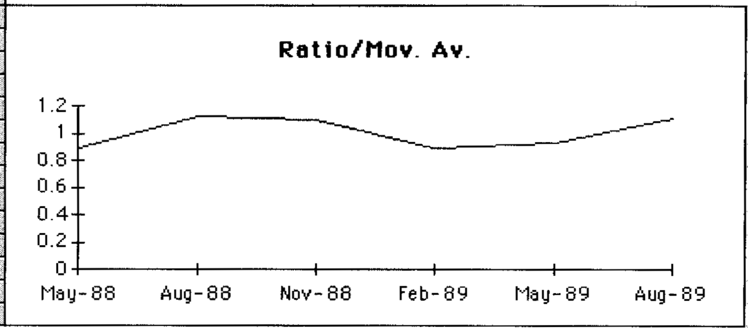
Date	Tonnes	Moving Aver	Ratio/Mov.Av.
Jan-88	45		
Apr-88	37		
Jul-88	50	49	0.8877551
Oct-88	64	51	1.1176471
Jan-89	53	53.25	1.0985916
Apr-89	46	55.25	0.8959276
Jul-89	58	56.25	0.9244444
Oct-89	68	56.75	1.1101322
Jan-90	55		

The above data was entered into a spreadsheet.

Date	Tonnes	Moving Aver.	Ratio/Mov.Av.
Jan-88	45		
Apr-88	37		
Jul-88	50	49	0.8877551
Oct-88	64	51	1.11764706
Jan-89	53	53.25	1.09859155
Apr-89	46	55.25	0.8959276
Jul-89	58	56.25	0.92444444
Oct-89	68	56.75	1.11013216
Jan-90	55		



Date	Ratio/Mov.Av.
May-88	0.8877551
Aug-88	1.11764706
Nov-88	1.09859155
Feb-89	0.8959276
May-89	0.92444444
Aug-89	1.11013216



- The cells in the moving average column on the same level as Jul-88 down to Oct-89 were 'selected'.
- The formula =(B4 + B5 + B6 + B7)/4 was used to automatically fill in the numbers in the 'moving averages' column.
- The graph immediately beneath the data was drawn using a 'chart'.
- The ratio/moving averages column was filled in using the formula =(B5 + B6)/(2 * C6).
- The second graph was drawn using another chart.

The whole process took five minutes.

The above calculation was carried out using an Excel spreadsheet. Some of the terminology may be slightly different for users of other spreadsheets.

Exercise 38c

1. Take the data from question 1 of exercise 38b and analyse it for periodicity.

2. Consider the following set of data:

Length	2	2	2	4	6	6	6	8
Date	Jan	Feb	March	April	May	June	July	Aug

Length	10	10	10	12
Date	Sept	Oct	Nov	Dec

   a. Graph the data.
   b. Calculate the moving averages of order 4.
   c. Graph the moving averages.
   d. Analyse the data for periodicity.

3. a. The following data is available:

Value	56	32	70	*a*	*b*	40	50	60
Month	Jan	Feb	March	April	May	June	July	Aug

The 4-point moving averages for this data are 63, 60.5, c, 57.5, d. Calculate the values of a, b, c, d.

Problems and Investigations

1. Obtain a time series graph such as that in example B of this chapter and investigate it thoroughly for secular, periodic and erratic variation trend.
2. Obtain data for some social statistic such as crime, unemployment, party preference and analyse it thoroughly using the techniques of this chapter. Comment on any erratic variations. Try to supply explanations for them.

39. PROBABILITY

Achievement Objectives

On completion of this chapter, students should be able, at:

LEVEL 6 STATISTICS

- *to use tables of multivariate data from social contexts to find the probabilities of everyday events or the proportion of outcomes in a given category*
- *to determine the theoretical probabilities of the outcomes of both exclusive and independent events such as rolling a die followed by the drawing of a card from a deck*
- *to use probability trees to calculate conditional probabilities*

Introduction

The **probability** of an event occurring is a measure of how certain or uncertain it is that the event will occur.

To estimate probabilities trials are performed. A trial is one repetition of an experiment. The outcomes are called **events**. The number of times the event occurs is observed and recorded.

The ratio $\frac{\text{number of occurrences}}{\text{number of trials}}$ estimates the probability of the event occurring.

Example A: To determine the probability of it raining on a day chosen at random during June, weather records for the past few decades would need to be examined and the number of days, n, on which it actually rained in June noted. If N was the total number of days in June checked, then $\frac{n}{N}$ would be a good approximation to the probability of rain falling on a randomly chosen day next June.

As the number of experiments performed increases, the accuracy of the probability estimate increases. Thus in example A, the probability of rain falling on a day in June could be estimated more accurately by studying records for several June months rather than just one.

The limit of the ratio $\frac{\text{number of occurrences}}{\text{number of trials}}$ as the number of trials is increased gives the probability of the event occurring.

The symbol $\hat{p}$ is used to represent an *estimate* of the probability from experiments.

Example B: To find the probability that a drawing pin tossed in the air will land on its side, a drawing pin is tossed a large number of times. A chart of the type below is filled in. n is the number of times the pin lands on its side and N is the total number of tosses. $\hat{p}$ is calculated by working out $\frac{n}{N}$.

n						
N	10	20	50	100	200	500
$\hat{p}$						

After 100 tosses the probability correct to at least one decimal place should be clear.

In a similar way, an estimate of the probability of many simple events such as obtaining two heads in three tosses of a coin, obtaining a sum of five by rolling two dice, having a spoon land face down when tossed, etc. can be established.

Exercise 39a

Devise procedures to find estimates of the probabilities of the following:

1. The next car that passes under the bridge being yellow in colour.
2. A person selected at random having a name beginning with L.
3. A female singing when you turn on the radio.
4. A dart thrown at a board hitting the bull's-eye.
5. The phone ringing any minute between 8.30 and 9.30 in an office.

Important Property of Probabilities

P(E) is the probability that event E occurs. If E is any event then $0 \le P(E) \le 1$ because P(E) is the limit of the ratio $\frac{\text{number of occurrences}}{\text{number of trials}}$ as N, the number of trials, gets larger. The smallest value, 0, will occur when the number of occurrences is 0 and the largest value, 1, will occur when the number of occurrences is N.
Thus $0 \le P(E) \le 1$.

Probabilities can be written as fractions, decimals or percentages.

Terminology

A **trial** is one repetition of an experiment. The **sample space** is a set whose elements describe every possible outcome of a trial.

Example C:

a. In an experiment a coin is to be tossed twice a number of times. The trial consists of tossing the coin twice and recording the outcome. The following are possible sample spaces.

i. {(H, H), (H, T), (T, H), (T, T)}, where H is a head and T a tail.
ii. {0, 1, 2} where each number is the possible number of tails.

b. If a netball team plays a series of games, the trial is the playing of a game and the recording of the result. The following are possible sample spaces:
i. {L, W, D} where the letters stand for 'Lost', 'Won', 'Drew'.
ii. {PB, PW} where PB stands for 'played badly', PW stands for 'played well'.

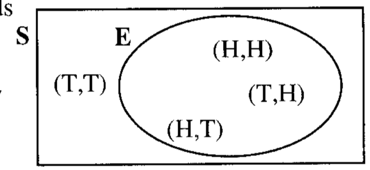
Equiprobable, or equally likely, outcomes occur when every element of a sample space has an *equal* likelihood of occurring.

Example D: From example C:

a. The probability of each element of {(H, H), (H, T), (T, H), (T, T)} would be found to be the same if two coins were tossed many times.
b. However, {0, 1, 2} is not a sample space of equiprobable outcomes. Probability experiments would show that 1 tended to occur twice as often as 0 or 2.

Mathematically an *event* is a subset of a sample space.

Example E: The event E, 'one or more heads in two tosses of a coin' is, when written as a subset of the sample space, {(H, H), (H, T), (T, H)}. This is illustrated by the **Venn diagram** of set theory.
S, the sample space, is the universal set.
E, the event, is the subset.



Finding Probabilities when all Outcomes are Equiprobable

When all possible outcomes of a trial have an equal chance of occurring (i.e. they are equiprobable) the probability of an event can be *calculated* rather than determined by performing experiments. The probability of an event E is usually written:

$$P(E) = \frac{\text{number of elements in E}}{\text{number of elements in S}} = \frac{n(E)}{n(S)}$$

The following examples are solved by establishing the full sample space and using the formula for P(E).

Example F: Find the probability that a person chosen at random was born on a day beginning with T.

Solution: Let S be the sample space.
S = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}
Let E be the event: Born on a day starting with T. E = {Tuesday, Thursday}

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{2}{7}$$

Example G: Find the probability that when a pair of dice are thrown, the two numbers will total four.

Solution: Letting S be sample space, we have:
S = {(6, 6), (6, 5), ..., (6, 1), (5, 6), (5, 5), ..., (1, 1)} which can be written in column form:

6,6	5,6	4,6	3,6	2,6	1,6
6,5	5,5	4,5	3,5	2,5	1,5
6,4	5,4	4,4	3,4	2,4	1,4
6,3	5,3	4,3	3,3	2,3	1,3
6,2	5,2	4,2	3,2	2,2	1,2
6,1	5,1	4,1	3,1	2,1	1,1

E = {(3, 1), (2, 2), (1, 3)}. The three ordered pairs (3, 1), (2, 2), (1, 3) are the only combinations of numbers on the two dice which give a sum of four.

$$\therefore \; P(E) = \frac{3}{36} \qquad [n(E) = 3, n(S) = 36 \text{ in formula}]$$

$$= \frac{1}{12}$$

Exercise 39b: Basic Probability

1. A marble is drawn at random from a bag which contains nine yellow and seven red marbles. Find:
   a. P (a yellow marble)
   b. P (a red marble)
   c. P (a yellow or red marble)
   d. P (neither a yellow nor a red marble)

2. If, in Question 1, a red marble is drawn and a second draw is made. Find:
   a. P (a yellow marble if the red one is put back into the bag)
   b. P (a yellow marble if the red one is *not* put back into the bag)
   c. P (a red marble if the first one is put back into the bag)
   d. P (a red marble if the first one is *not* put back into the bag)

3. A marble is drawn at random from a bag which contains eight red, five white and seven blue marbles. Find:
   a. P (a red marble) b. P (a white marble)
   c. P (a blue marble) d. P (*not* a red marble)

4. If in Question 3, a blue marble is drawn, and a second draw is made. Find:
   a. P (a white marble if the blue one is put back into the bag)
   b. P (a white marble if the blue one is *not* put back into the bag)
   c. P (a blue marble if the first one is put back into the bag)
   d. P (a blue marble if the first one is *not* put back into the bag)

5. Find the probability that a day of the week drawn at random will:
   a. start with the letter S b. start with the letter W
   c. start with the letters W or T d. *not* start with the letters W or T
   e. end with the letter Y f. *not* end with the letter Y

6. Find the probability that a month of the year drawn at random will:
   a. have more than 30 days b. have less than 30 days
   c. start with the letter M d. end with the letter Y
   e. start with the letter J *and* end with the letter Y
   f. start with the letter J *or* end with the letter Y
   (Note: "or" includes both, e.g. January)

7. Find the probability that a number drawn at random from {1, 2, 3, 4, 5, 6, 7, 8, 9} will be:
   a. even b. odd
   c. either even or odd d. neither even nor odd
   e. a multiple of 3 f. a factor of 4
   g. a multiple of 3 *and* a factor of 3
   h. a multiple of 3 *or* a factor of 6

8. Find the probability that a card drawn at random from a pack of 52 cards is:
   a. an ace b. a heart
   c. the ace of clubs d. an ace or a heart
   e. neither an ace nor a club f. an ace but *not* a spade
   g. a spade but *not* an ace

9. Find the probability that a card drawn at random from a pack of 52 cards is:
   a. a red card
   b. a picture card (does not include aces)
   c. a red picture card
   d. a red card or a picture card
   e. neither a red card nor a black card
   f. a red card but *not* a picture card
   g. a picture card but *not* a red card

10. A jar contains three blue marbles and two white marbles.
    a. One marble is randomly withdrawn and its colour noted.
       i. Write down the sample space of outcomes for this experiment.
       ii. By using your answer to i. find the probability of withdrawing a white marble.
       iii. What is the probability of withdrawing a blue marble?

b. One marble is randomly withdrawn and its colour noted. It is replaced and another is randomly withdrawn with its colour being noted.
   i. Write down the sample space of outcomes for this experiment.
   ii. Use your answer to i. to find the probability of both marbles being blue.
   iii. Find the probability of getting blue on the first marble and white on the second.
   iv. Find the probability of getting a blue and a white marble in the two draws, not necessarily in that order.

c. One marble is randomly withdrawn with its colour being noted. Without replacing this marble another is withdrawn and the colour is noted.
   i. Write down the sample space for this experiment.
   ii. Find the probability of drawing two white marbles.
   iii. Find the probability of drawing a blue and a white in that order.
   iv. Find the probability of drawing at least one white.
   v. Find the probability of drawing a blue on the second draw.

11. A box contains two blue balls, two red balls and one white ball.
   a. One ball is randomly withdrawn and its colour noted.
      i. Write down the sample space of outcomes for this experiment.
      ii. By using your answer to i. find the probability of withdrawing the white ball.
      iii. What is the probability of not drawing a red ball?
   b. One ball is randomly withdrawn and its colour noted. It is replaced and another one is randomly withdrawn and its colour noted.
      i. Write down the sample space of outcomes for this experiment.
      ii. Use your answer to i. to find the probability of getting two blue balls.
      iii. Find the probability of getting no blue balls.
      iv. Find the probability of drawing a red ball on the second draw.
   c. One ball is randomly withdrawn and its colour noted. Without replacing this ball another is withdrawn, and its colour noted.
      i. Write down the sample space for this experiment.
      ii. Find the probability of withdrawing blue then red in that order.
      iii. Find the probability of withdrawing two balls of the same colour.
      iv. Find the probability of getting the white ball in one of the two draws.
      v. Find the probability of not drawing the white ball.

12. In a room there are four men and two women.
   a. One person is randomly withdrawn and the sex of the person noted.
      i. Write down the sample space of outcomes for this experiment.
      ii. By using your answer to i. find the probability of a woman being withdrawn.
      iii. What is the probability of not withdrawing a man?

   b. One person is randomly withdrawn. The person returns and another is randomly withdrawn.
      i. Find the probability of getting two people of a different sex.
      ii. Find the probability of getting at least one man.
      iii. Find the probability of the first person leaving being male.
   c. One person is randomly withdrawn and the person's sex is noted. Without the person returning another is withdrawn, and their sex noted.
      i. Find the probability of more than one man leaving.
      ii. Find the probability of no women leaving.
      iii. Find the probability of the second person who leaves being a woman.
      iv. Find the probability of both people leaving being women.

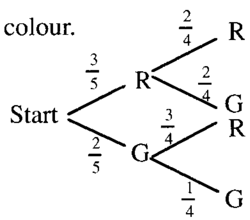
Probability Trees

The process of writing out sample spaces as in the previous two examples always works, but can become almost impossible when the sample spaces become too large.

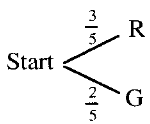
In such circumstances, probability problems are tackled by construction of a **probability tree**. A probability tree follows an experiment or process through its various stages with probabilities being calculated at each stage.

Example H: A jar contains three red balls and two green balls. One ball is withdrawn and then another, without the first one being replaced.
a. Find the probability that both balls are red.
b. Find the probability that the balls are different in colour.

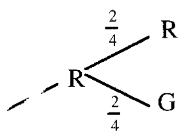
Solution: The probability tree for withdrawing one ball then another without replacing the first is:



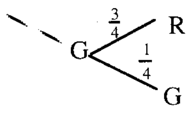
It is drawn as follows:



In the first withdrawal there is a $\frac{3}{5}$ chance of removing a red ball and a $\frac{2}{5}$ chance of removing a green ball.



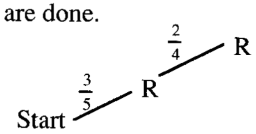
If a red ball had been removed initially there would be two reds and two greens in the jar, leaving in the second withdrawal a $\frac{2}{4}$ chance of removing a red ball and a $\frac{2}{4}$ chance of removing a green ball.



Similarly, if a green ball had been removed initially, there are now three reds and one green in the jar, leaving in the second withdrawal a $\frac{3}{4}$ chance of removing a red ball and a $\frac{1}{4}$ chance of removing a green ball.

To complete the problem, some simple calculations are done.

a. The probability of two red balls is found by multiplying all fractions on the branches:



The probability of two red balls = P(R, R)

$= \frac{3}{5} \times \frac{2}{4}$

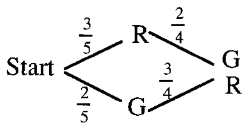
$= \frac{3}{10}$

b. Different coloured balls on each draw means either a red followed by a green ball or a green followed by a red ball. The probability of 'different coloured balls on each draw' = P(R, G) + P(G, R)

$= \frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{3}{4}$

$= \frac{12}{20}$

$= \frac{3}{5}$

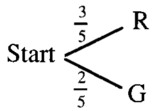


Note:

a. (R, G) and (G, R) are different events and should not be confused.

b. The sum of probabilities for each set of branches from a point is 1.

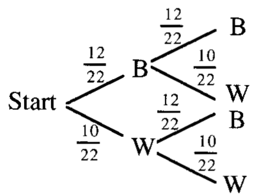
e.g.



$\frac{3}{5} + \frac{2}{5} = 1$

Example I: A bag contains twelve black cards and ten white cards. A card is withdrawn and then replaced and another withdrawn. Find the probability, using a probability tree, that both cards are the same colour.

Solution: The probability tree is:



P (two cards the same)

$= P(B,B) + P(W,W)$

$= \frac{12}{22} \times \frac{12}{22} + \frac{10}{22} \times \frac{10}{22}$

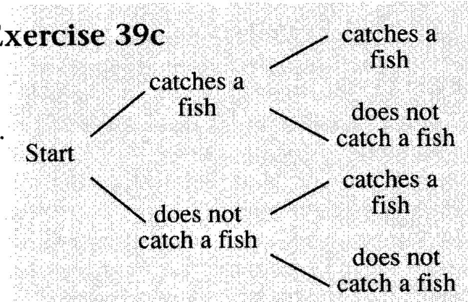
$= \frac{244}{484}$

$= \frac{61}{121}$

Note: In the second stage (right) branches, the probabilities of getting a black or a white card remain the same because the original card withdrawn was replaced.

Exercise 39c

1.



The above tree diagram describes the possible outcomes for a person who goes fishing two days in a row.

a. If the probability of catching a fish is always $\frac{1}{3}$ whenever they go fishing, what is the probability that:
   i. they don't catch a fish on either day.
   ii. they are successful on the first day and not the second day.
   iii. they catch a fish on at least one day.

b. Answer the same question as for (a) with the following changes. The probability of success is $\frac{1}{3}$ on the first day. If they are successful on the first day their chance of success on the second day is $\frac{3}{4}$. If they are unsuccessful on the first day their chance for success on the second day sinks to $\frac{1}{4}$.

2. A box contains equal numbers of 10 cent and 5 cent pieces. A coin is randomly withdrawn from the box and replaced. This is done three times.
   a. Draw a tree diagram for this.
   b. Use your answer to a. to answer the following:
      i. What is the probability of getting three 10 cent pieces?
      ii. What is the probability of two 5 cent pieces and one 10 cent piece in the three draws?
      iii. What is the probability of getting at least one 5 cent piece?
      iv. What is the probability of getting a 10 cent piece on each of the last two draws?
      v. What is the probability of not getting three 5 cent pieces?

3. A lazy 6th former has a three question test which he has not studied for. Each answer is either true or false. He guesses the answer for each question.
   a. Draw a tree diagram for this.
   b. Use your answer to a. to answer the following:
      i. What is the probability that he got all three questions right?
      ii. What is the probability that he got all the questions wrong?
      iii. What is the probability, if the correct answers were T, F, F, that he got the last two right?
      iv. What is the probability that he will fail the test if a fail is less than half right?

4. A tourist travels to a tropical country. There are two types of mosquito in this country, type A and type B. Type A is twice as common as type B. The tourist gets bitten twice in one night.
   a. What is the probability that she was bitten by type A on each occasion?
   b. What is the probability that type B bit her first?
   c. What is the probability that she was bitten at least once by type B?
   d. What is the probability that she was bitten by type A first?

5. A box contains twelve blue marbles, ten red marbles and five white marbles. A marble is withdrawn, and without replacing it another is withdrawn.
   a. What is the probability that at least one of the marbles was white?
   b. What is the probability that both marbles were blue?
   c. What is the probability that the marbles did not include a red one?
   d. What is the probability that at least one of the marbles was blue?
   e. What is the probability that the second marble withdrawn was blue?

6. A box contains thirty red cards, twenty blue cards and ten white cards. Two are withdrawn, one after the other without replacement.
   a. What is the probability that the first card withdrawn is blue?
   b. What is the probability that a white card is not withdrawn?
   c. What is the probability that the second card withdrawn is red?
   d. What is the probability that a white and red are withdrawn?
   e. What is the probability that both cards are red?

7. People are waiting at a bus stop in a line. Two are French, one is German, and the other is an Arab.
   a. What is the probability that the first person to get on the bus is Arabian?
   b. What is the probability that the second person to get on the bus is French?
   c. What is the probability that neither of the first two people to get on the bus is French?
   d. What is the probability that the German is among the last two people to get on the bus?
   e. What is the probability that the first three people to get on the bus are French, Arabian and German, not necessarily in that order?

Mutually Exclusive Events

Two events, E_1 and E_2, are **mutually exclusive** if it is impossible for them to happen together.

Example J:

a. The two events 'Shane has blond hair' and 'Shane is bald' are mutually exclusive.

b. A coin is tossed twice. E_1 is the event 'First throw is heads', and E_2 is the event 'First throw is tails'. E_1 and E_2 are mutually exclusive.

If A and B are mutually exclusive events then the probability of either of them happening is the sum of their probabilities.

Example K: The probability of a person not have a driving license is 0.23. The probability of a person being a registered taxi driver is 0.003. What is the probability of a randomly selected person being either a taxi driver or not having a driving license?

Solution: Quite clearly someone cannot be a registered taxi driver and not have a license at the same time. The two events are mutually exclusive, ie: they cannot happen together.

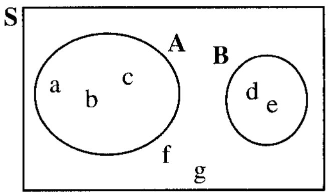
∴ probability of either of them occurring is their sum which is 0.233.

Generally, if A and B are mutually exclusive events, then A and B have no common elements; their **intersection** is the **empty set**, $\mathbf{A \cap B = \emptyset}$.

If A and B are mutually exclusive:

a. The probability of A *and* B occurring is zero. i.e. $P(A \cap B) = 0$.

b. The probability of A *or* B occurring is the probability of A plus the probability of B. i.e. $P(A \cup B) = P(A) + P(B)$.



The Venn diagram illustrates the situation which occurs when two events, A and B, are mutually exclusive. Note that they have no part of the sample space in common.

In this case, A = {a, b, c}, B = {d, e} and S, the sample space, = {a, b, c, d, e, f, g}.

Complementary Events

Two events are **complementary** if:

a. they are mutually exclusive, and

b. every possible outcome of a trial belongs in one of the events.

In other words joining the two events gives the sample space.

Example L: Suppose a person is being selected at random from a list of names. If M = 'person selected is male'
F = 'person selected is female'
then quite clearly M and F are complementary events.

If A and B are complementary events:

i. $\mathbf{P(A \cup B) = 1}$. This is because $A \cup B$ is the sample space and the probability of an outcome being in the sample space is 1.

ii. $\mathbf{P(A) + P(B) = 1}$. This follows because A and B are mutually exclusive.

iii. If A and B are complementary events B is often given the symbol A′. This follows the similar notation used in set theory for the complement of a set. Similarly, A could be given the symbol B′.

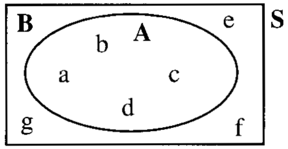
A′, the complement of A, is the event "A does not happen".

The following Venn diagram illustrates two complementary events, A and B.

S = {a, b, c, d, e, f, g}

A = {a, b, c, d}

B = A′ = {e, f, g}



Exercise 39d

1. A survey about unemployment was taken from the voters in a suburb. The results are shown below.

	Party X	Party Y	Party Z
Employed	47	53	21
Unemployed	13	7	19

   a. How may of the voters from Party Y are employed?
   b. How many of the unemployed prefer Party X?
   c. What is the total number of people surveyed?
   d. What is the number of people surveyed who are employed?
   e. What is the probability a randomly selected person is employed?
   f. What is the probability a randomly selected person prefers Party Y?
   g. What is the probability a randomly selected voter prefers Party Z?
   h. What is the probability a randomly selected voter is unemployed?
   i. What is the probability a randomly selected voter prefers Party Z and is employed?
   j. What is the probability a randomly selected voter is employed or prefers Party Y?
   k. Explain why 'preferring Party X' and 'preferring Party Y' are mutually exclusive.
   l. Explain why 'preferring Party Z' and 'being unemployed' are not mutually exclusive.
   m. What event is complementary to 'being employed'?
   n. What event is complementary to 'belonging to Party X or Y'?
   o. What event is complementary to 'belonging to Party X'?
   p. What is the probability a person is either 'employed and prefers Party X' or 'unemployed and prefers Party Y'?

2. A coin is flipped twice. Write down which of the following pairs of events are mutually exclusive.
   a. (Head on first throw), (tail on second throw)
   b. (Head on first throw), (tail on first throw)
   c. (Head on first throw, tail on second), (tail on first throw, head on second)
   d. (Head on first throw), (head on second throw)

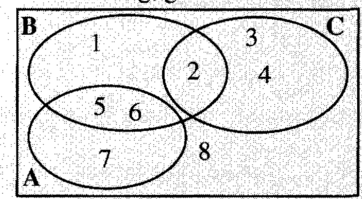
3. The probability of a randomly selected person being a bald male is 0.025. The probability of a randomly selected person being a bald female is 0.0037.
   a. What is the probability a randomly selected person is not a bald male?
   b. What is the probability a randomly selected person is bald?
   c. What is the probability a randomly selected person is not bald?
   d. What is the probability that if you selected two people randomly they would both be bald?

4. An experiment has the sample space {1, 2, 3, 4, 5, 6, 7, 8} of equally likely outcomes. Write down the events which are complementary to:
   a. {1, 2} b. {3, 4, 5}
   c. {factors of 64 less than 11} d. {multiples of 3 less than 9}
   e. {1, 2, 3, 4, 5, 6, 7, 8}

5. $E_1 = \{1, 2\}$ $E_2 = \{2, 3\}$ $E_3 = \{4, 5, 6\}$ $E_4 = \{1, 4, 5, 6, 7, 8\}$
   $E_5 = \{1, 2, 3, 7, 8\}$ are events from the sample space {1, 2, 3, 4, 5, 6, 7, 8}
   a. Which pairs of events are mutually exclusive?
   b. Which pairs of events are complementary?
   c. Work out the following probabilities, where all outcomes are equiprobable:

i. $P(E_1)$	ii. $P(E_2)$	iii. $P(E_3)$
iv. $P(E_4)$	v. $P(E_5)$	vi. $P(E_1 \cup E_2)$
vii. $P(E_1 \cup E_4)$	viii. $P(E_2 \cap E_3)$	ix. $P(E_4 \cap E_5)$
x. $P(E_1')$	xi. $P(E_3')$	xii. $P(E_1 \cup E_2')$
xiii. $P(E_1' \cup E_3')$	xiv. $P(E_2 \cap E_5')$	xv. $P((E_1 \cup E_2) \cap E_5)$

6. Use the Venn diagram to work out the following, given that each outcome is equally likely:
   a. $P(A \cup B)$
   b. $P(A) + P(B) - P(A \cap B)$
   c. $P(B \cup C)$
   d. $P(B) + P(C) - P(B \cap C)$
   e. $P(A \cup C)$
   f. $P(A) + P(C) - P(A \cap C)$
   g. Generalise any observation you might have made in a. to f. to any pair of events E and F by writing down an expression equal in value to $P(E \cup F)$.



7. A, B, C are mutually exclusive events: $P(A) = \frac{1}{3}$ $P(B) = \frac{1}{4}$ $P(C) = \frac{1}{5}$
   a. Work out the following probabilities:

i. $P(A')$	ii. $P(A \cup B)$	iii. $P(B')$
iv. $P(A \cup C)$	v. $P(A \cap B)$	vi. $P(A \cup C')$
vii. $P(A \cap C)$	viii. $P(C')$	ix. $P(C' \cap A)$
x. $P(B' \cap C)$		

   b. What is the smallest possible number of elements in the sample space?
   c. Why are A and B not complementary events?

Independent Events

Two events, A and B, are **independent** if the outcome of one has absolutely no effect on the outcome of the other.

Example M:

a. Suppose a dice is being rolled and a coin is tossed. If A is 'a number less than 3 occurs' and B is 'heads', then A and B are completely independent.

b. Joe lives in the USA; Joan lives in the UK; neither knows the other.
E is the event: 'Joe eats cornflakes on July 14'
F is the event: 'Joan eats muesli on July 14'.
E and F are obviously independent events except in the most contrived circumstances.

An important property of independent events is that the probability of their both occurring is the product of their probabilities.

Example N: A is the event that a number less than 3 occurs on the first roll of a dice. B is the event that a number greater than 3 occurs on the second roll of a dice. Find the probability that both A and B occur if a dice is rolled twice.

Solution: $P(A) = \frac{1}{3}$ $\qquad$ $P(B) = \frac{1}{2}$

A and B are independent. The event of a number less than 3 occurring on the first roll will have absolutely no effect on whether a number greater than 3 occurs on the second roll.

$\therefore$ probability of both A and B occurring is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

[The probability of both A and B occurring is often written $P(A \cap B)$]

Note: Take care not to confuse 'mutually exclusive events' and 'independent events'. They are two *totally different* concepts.

Exercise 39e

1. A bag contains 2 blue balls and 3 red balls. A ball is removed, its colour noted then replaced. This is repeated once. Which of the following events are INDEPENDENT?
   a. (First ball is blue), (second is red)
   b. (First ball is blue), (second is blue)
   c. (First ball is blue), (first ball is red)
   d. (First ball is blue, second is red), (second ball is blue)
   e. (First ball is blue, second is red), (second ball is red)

2. Answer the questions from (1) if the first ball removed is not replaced.

3. For each of the following pairs of events write down a situation in which they would be independent and a situation when they would not be independent.
   a. James is tall, Dean is short
   b. Sally is happy, Samuel is sad
   c. The dog wags his tail, Dave has a bone
   d. Susan passes her exam, Sarah passes her exam
   e. It rains on Saturday, it rains on Sunday

4. a. If you select a random 2 digit number, which of the following pairs of events are independent?
      i. First digit less than 3, second digit greater than 7
      ii. First digit is prime, second digit is prime
      iii. First digit is less than 4, sum of digits is 7
      iv. First digit is a factor of the second, second digit is greater than 5
   b. For each pair of events which are independent find the probability of both occurring.

5. The probability of event A is 0.6. The probability of event B is 0.43. The probability of events A *and* B occurring is 0.37. Explain why A and B are not independent.

6. A jar contains two black marbles and three red marbles. A marble is removed and its colour noted. It is then replaced and this is repeated.
   E_1 is the event "First marble is black"
   E_2 is the event "Second marble is black"
   E_3 is the event "First marble is red"
   E_4 is the event "Second marble is red"
   E_5 is the event "First marble is black, second marble is red"

   a. From the above events, list those which are mutually exclusive, complementary or independent.
   b. Write out each of the following events in words:

i. E_1'	ii. E_2'	iii. $E_1 \cap E_2'$
iv. $E_3' \cap E_4'$	v. E_5'	

c. Evaluate the following probabilities:

i. $P(E_1)$	ii. $P(E_2)$	iii. $P(E_3)$
iv. $P(E_4)$	v. $P(E_5)$	vi. $P(E_1 \cup E_2)$
vii. $P(E_1 \cap E_2)$	viii. $P(E_2 \cup E_3)$	ix. $P(E_2 \cap E_3)$
x. $P(E_1 \cap E_3)$	xi. $P(E_1 \cup E_3)$	xii. $P(E_4 \cap E_5)$
xiii. $P(E_3 \cup E_5)$	xiv. $P(E_1 \cup E_5)$	xv. $P(E_1')$
xvi. $P(E_1' \cap E_2)$	xvii. $P(E_2' \cap E_3)$	xviii. $P(E_1 \cup E_2')$
xix. $P(E_5' \cap E_1)$	xx. $P(E_2 \cap E_4')$	

Problems and Investigations

Investigate the election of a sub-committee of three people from a group of eight consisting of four married couples. What will be the probabilities of:

1. A married couple on the sub-committee
2. Three people of the same sex on the committee
3. Three people not related to each other on the sub-committee

40. CONDITIONAL PROBABILITY

Achievement Objectives

On completion of this chapter, students should be able, at:

LEVEL 6 STATISTICS

- *to use probability trees to calculate conditional probabilities*

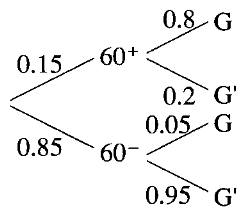
Introduction

Suppose you were to ask a person what the chance would be of getting someone with grey hair if you picked them randomly from the population as a whole. The chances suggested would probably vary between 5% and 20% as most people appreciate that only a small minority of the population has grey hair. However, what if you asked the probability of picking someone with grey hair if you restrict yourself to those over 60 years of age? Then the answer would be quite high, probably 80% to 95%.

This is a good example of the idea of **conditional probability**, where you alter the probability by restricting your sampling to a subset of the original population.

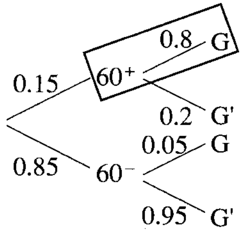
Example A: Assume that 15% of the population is over 60. Assume that 80% of those over 60 have grey hair. Assume that 5% of those under 60 have grey hair. Represent this on a tree diagram.

Solution: Let the event 'someone is over 60' be represented by 60^+. The complementary event is 'someone is less than 60'. Represent this by 60^-. Let the event 'has grey hair' be G. The complementary event is G' read as 'does not have grey hair'. The tree appears below.



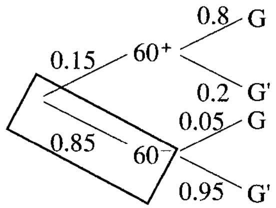
In the following, consider those parts of this tree selected and framed.

a.



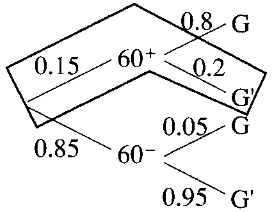
This says the probability of 'someone having grey hair if they are 60 or more' is 0.8.

b.



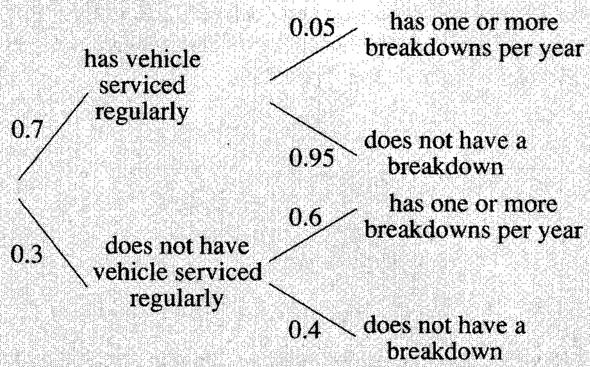
This says the probability of 'someone being under 60' is 0.85.

c.



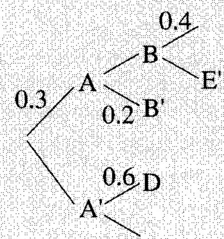
This says the probability of a randomly selected person 'being 60 or more and not having grey hair' is $0.15 \times 0.2 = 0.03$.

Exercise 40a



This tree shows the results of a recent survey of users of a certain type of car. Answer the following questions.

1. What is the probability someone with this sort of car does not have their car serviced regularly?
2. What is the probability someone with this sort of car does not have a breakdown during the year if they have the vehicle serviced regularly?
3. What is the probability someone with this sort of car has one or more breakdowns during the year if they do not have their car serviced regularly?
4. What is the probability someone with this sort of car has their vehicle serviced regularly and has had one or more breakdowns during the year?

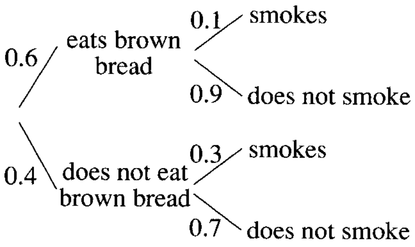


5. Copy this tree diagram and put on the six missing pieces of information referring to this diagram.
6. What is the probability of D' occurring if A' has occurred?
7. What is the probability of E occurring if A and B have occurred?
8. What is the probability of A' and D both occurring?
9. What is the probability of A, B, and E' all occurring?
10. In a school all students study maths and physics. The probability of a student passing the maths exam is 0.7. If a student passes the maths exam the probability of passing the physics exam is 0.8. If a student fails the maths exam the probability of passing the physics exam is 0.1.
    a. Draw a probability tree showing this information.
    b. What is the probability a randomly selected student passes physics?

Symbol for Conditional Probability

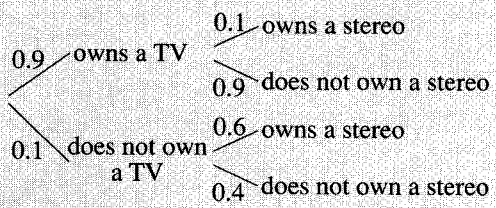
The symbolic way of writing the probability that B occurs if A has occurred is $P(B \mid A)$.

Example B:



This tree shows the results of a survey of young people in a town. The probability that such a person smokes if they eat brown bread is 0.1. Symbolically this would be written P (smokes | eats brown bread) = 0.1

Exercise 40b



Questions 1-5 refer to the above diagram. Find the following:

1. P (owns a stereo | owns a TV)
2. P (does not own a stereo | does not own a TV)
3. P (owns a TV)
4. P (owns a TV and does not own a stereo)
5. P (owns a stereo and does not own a TV)

A survey of computer ownership among pupils at a school revealed the following:

	Apple	Acorn	PC	Other	Total
Junior	25	10	32	33	100
Senior	40	8	30	22	
Total					

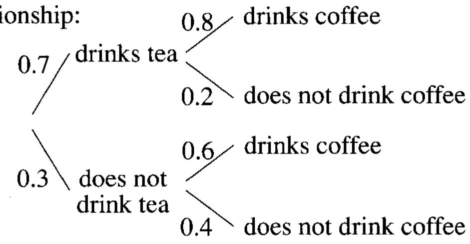
6. Copy and complete the table.

Using this table calculate the following based on a randomly selected computer owner from this school.

7. Probability of owning an Apple Computer.
8. Probability of owning a PC if a senior.
9. P (Acorn | Junior)
10. P (Junior | Other)

Formula for Conditional Probability

An important relationship:



Consider the above diagram. The probability of someone drinking tea and drinking coffee is 0.7×0.8. This is found by multiplying along the path that links 'drinks tea' with 'drinks coffee'.

Thus: P (Drinks tea and drinks coffee) = 0.56
P (Drinks tea $\cap$ drinks coffee) = 0.56
P (Drinks tea) $\times$ P (drinks coffee | drinks tea) = $0.7 \times 0.8 = 0.56$
P (Drinks tea) $\times$ P (drinks coffee | drinks tea) = P (drinks tea $\cap$ drinks coffee)

This can be generalised to any pair of events by the important relationships;

$$\mathbf{P(A \cap B) = P(B)\,P(A \mid B)}$$

which is equivalent to;

$$\mathbf{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

Example C: Referring to the 'tea and coffee' probability tree find the following (note: 'someone' refers to a randomly chosen person).

i. Probability that someone does not drink coffee if they drink tea.
ii. P (drinks coffee | does not drink tea).
iii. Probability someone does not drink tea and does not drink coffee.
iv. P (drinks tea $\cap$ does not drink coffee).
v. Probability someone drinks coffee.
vi. Probability someone drinks tea if they drink coffee.
vii. P (does not drink tea | does not drink coffee)

Solution: From the diagram:

i. Probability that someone does not drink coffee if they drink tea is 0.2.
ii. P (drinks coffee | does not drink tea) = 0.6.
iii. Probability someone does not drink tea and does not drink coffee = $0.3 \times 0.4 = 0.12$.

iv. P (drinks tea ∩ does not drink coffee) = 0.7 × 0.2 = 0.14

v. Probability someone drinks coffee is calculated by multiplying the probabilities along all the routes that lead to coffee then adding them.
∴ P (someone drinks coffee)
$= 0.7 \times 0.8 + 0.3 \times 0.6$
$= 0.74$

vi. P (drinks tea | drinks coffee)

$$= \frac{\text{P (drinks tea and drinks coffee)}}{\text{P (drinks coffee)}}$$

$$= \frac{0.7 \times 0.8}{0.74} \quad \text{[from part v]}$$

$$= \frac{56}{74}$$

vii. P (does not drink tea | does not drink coffee)

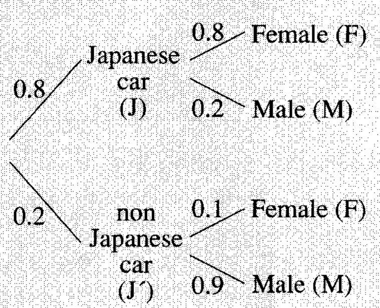
$$= \frac{\text{P (does not drink tea} \cap \text{does not drink coffee)}}{\text{P (does not drink coffee)}}$$

$$= \frac{0.3 \times 0.4}{1 - 0.74} \quad \text{[complementary events]}$$

$$= \frac{0.12}{0.26}$$

$$= \frac{12}{26}$$

Exercise 40c



Questions 1-6 refer to the above diagram which arose from a survey of car ownership.

Calculate:

1. P (F | J) 2. P (M | J′) 3. P (J′ ∩ F) 4. P (M)
5. P (J | M) 6. P (J′ | F)

7. The probability of being left-handed and playing hockey is 0.005. The probability of playing hockey is 0.1. What is the probability of being left-handed if you play hockey?

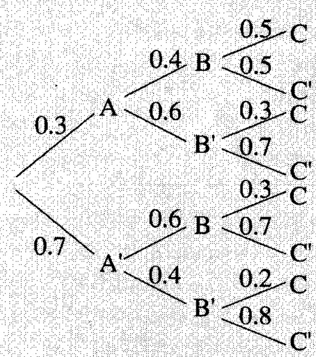
8. The probability of owning an encyclopedia if you own a dictionary is 0.25. The probability of owning a dictionary is 0.93.
   a. Find the probability of owning a dictionary and an encyclopedia.
   b. If the probability of owning an encyclopedia is 0.36 find the probability of owning a dictionary if you own an encyclopedia.

9. The probability of studying maths if you study chemistry is 0.85. The probability of studying chemistry is 0.60.
   a. Find the probability of studying chemistry and maths.
   b. If the probability of studying maths is 0.88 find the probability of studying chemistry if you study maths.

10. The probability of playing soccer and cricket is 0.32. The probability of playing soccer and not playing cricket is 0.13.
   a. What is the probability of playing soccer?
   b. What is the probability of playing cricket if you play soccer?
   c. If the probability of not playing cricket is 0.31 what is the probability of playing soccer if you play cricket?

Problems and Investigations

1.



Find:
a. P (C)
b. P (B | C)
c. P (A ∪ C | B)

2. A, B, C, D compete in a knockout tournament. To win the tournament A must first defeat B and then defeat the winner of the match between C and D.
   A has probabilities of $\frac{1}{3}$, $\frac{5}{7}$ and $\frac{4}{5}$ of defeating B, C and D respectively, while C has $\frac{3}{7}$ chance of beating D.
   a. If A has already defeated B but does not know the outcome of the match between C and D, what is the probability A will win the tournament?
   b. If no assumption is made about the outcome of the match between A and B and if A did not win the tournament, what is the probability C defeated D in the first round?

41. RANDOM NUMBERS AND SIMULATIONS

Achievement Objectives

On completion of this chapter, students should be able, at:

LEVEL 7 STATISTICS

- *to simulate situations using dice or random number generators to calculate probabilities of outcomes*

Random Numbers

These are sequences of numbers where each digit in every number has been selected randomly from {0, 1, 2, 3, ..., 8, 9}. This means that every digit is completely independent in the statistical sense from its predecessors and successors. The numbers in random number tables are genuinely random. Some random number generators on calculators and computers are pseudo-random in that the same set of numbers is generated repeatedly. This is usually a very large set so that no problem should arise.

Example A: Random digits are being produced by a computer.

i. What is the probability the next digit will be either 1, 2, 3.

Solution: 0.3 [as there are ten possible outcomes namely 0, 1, 2, ..., 9]

ii. If the last seven digits have been 9's what is the probability that the next digit will be a 9?

Solution: 0.1 [Even though the last seven digits have been 9's there is still only a 1 in 10 chance the next digit will be a 9 (unless there is something wrong with the computer).]

iii. Trios of the randomly generated digits are being produced. What is the probability the first and third digits in a trio are less than 4 and the middle digit is bigger than 3?

Solution: Types of trios which would be acceptable are 152, 340, 091, ... Types which would not be acceptable are 876, 112, 903, ...

If you are unable to calculate this using probability theory you could get a pretty close estimate by writing down one hundred trios and counting those trios which are acceptable. Using probability theory we note the following:

The probability of a digit being less than 4 is 0.4.
The probability of a digit being greater than 3 is 0.6.

As the individual digits of these trios are **independent** the probability of (1st less than 4), (2nd greater than 3), (3rd less than 4) is $0.4 \times 0.6 \times 0.4 = 0.096$. [Probabilities are multiplied as the three events are independent.]

Note: if you have attempted to solve this problem by counting the number of satisfactory trios out of a large set you would have probably found that your probability would have been close to 0.1.

Exercise 41a

Students can attempt these problems in two different ways:

i. by creating a large set of the type of random numbers specified then counting those that meet the criteria;
ii. by use of probability theory from the previous chapter.

1. Three digit numbers are randomly generated using the digits 0, 1 and 2. Find the following:
   a. The probability of getting all three digits being 0.
   b. The probability of getting exactly two 1s.
   c. The probability of getting at least one 1 among the three digits.
   d. The probability of getting a sum of four with the three digits.
   e. The probability of the number being even.
   f. The probability of the number being divisible by three.
   g. The probability of the number being divisible by ten.
   h. The probability of the last digit being a 3.
   i. The probability of having less than three 1s.
   j. The probability of the last digit being a 2 if the previous two digits were not.

2. Random numbers with two digits are being used. One is picked at random. (00, 01, . . . , 99 are possible.)
   a. What is the probability that it finishes with a zero?
   b. What is the probability that both digits are less than four?
   c. What is the probability that the sum of both digits is 12?
   d. What is the probability that both digits are even?
   e. What is the probability that the first digit is less than the second?

3. Single digits are being generated randomly.
   a. What is the probability that the first two digits are both 1?
   b. What is the probability the first two digits are from the set 0 and 1?
   c. If the first three digits are 3, what is the probability that the next digit is a 3?
   d. If the first four digits are all nines, what is the probability that the next three are fours?
   e. What is the probability that all three of the first digits are different?

Simulating Events using Random Numbers

It is possible to get a good estimate of success or failure of a particular course of action by constructing a mathematical model of what is proposed and finding out how often satisfactory outcomes occur.

In a major war during the 1970s one side decided to quit the conflict because analysts created a mathematical model with computers of the conflict which demonstrated that no matter what course of action was taken, victory was impossible.

Example B: A group of five people come into a bank. They enter randomly and are unknown to each other. What is the probability that exactly two are males?

Solution: Although this can be done using the methods of the previous chapter these are somewhat untidy. A good estimate can be found using random number simulations.

M	F	M	M	F	M	F	F	F	F	F	M	F	M	M
F	F	F	F	F	M	F	M	F	F	F	M	M	F	M
F	M	F	F	M	F	F	M	F	M	F	M	F	M	M
M	M	F	F	F	M	F	F	M	F	F	F	F	M	M
M	F	F	F	M	F	M	F	M	M	M	F	F	F	F
F	F	M	F	F	M	M	M	M	M	M	F	M	M	M
M	F	F	M	F	M	F	F	M	F	F	M	F	M	F
F	F	M	M	F	F	F	F	F	F	F	F	M	M	M
M	M	F	M	F	F	M	M	F	M	F	M	F	F	F
F	F	M	M	M	F	M	F	F	F	M	F	M	M	M
F	M	F	F	M	F	F	M	F	M	F	F	M	F	M
M	F	M	F	M	F	M	F	F	F	F	F	M	M	F
F	M	F	F	M	F	F	F	M	M	M	M	F	M	M
F	M	F	F	F	M	M	F	F	F	F	F	F	M	M
F	F	M	M	F	F	F	F	F	F	M	M	F	M	M
M	M	M	F	M	F	M	F	F	F	M	M	M	M	F
M	M	F	M	F	F	M	F	F	F	M	M	F	M	M
M	F	F	M	F	F	M	F	F	M	F	F	F	M	F

F	M	F	M	F	M	M	F	F	M	M	M	M	F	M
M	M	F	F	M	F	M	F	M	M	F	M	F	M	F
M	M	M	M	F	F	F	M	M	M	F	M	M	M	F
M	F	F	F	M	M	F	F	F	F	F	F	M	F	F
M	M	M	M	M	M	F	M	F	F	F	M	M	M	F
M	F	F	M	F	M	F	F	M	F	M	F	F	F	F
F	F	F	F	F	F	M	F	F	M	M	F	M	M	M
M	M	F	M	M	F	F	M	M	M	F	F	F	M	F
M	F	M	F	M	F	F	M	M	M	M	M	F	M	F
F	M	F	M	M	F	F	F	F	M	F	M	M	F	F
M	F	M	F	M	M	M	F	M	M	M	M	F	F	M
F	F	F	M	F	M	F	F	F	F	M	M	F	F	F
F	M	F	M	F	M	F	F	F	F	F	M	M	F	M
F	F	M	M	F	M	M	F	F	F	F	F	F	F	F
F	M	F	M	F	M	M	F	F	F	M	F	F	F	F
F	F	M	F	F	M	M	M	F	M	M	F	F	F	F
M	M	M	F	F	F	F	F	M	M	M	F	M	F	M
M	M	F	F	F	M	F	F	M	M	M	M	M	M	M
F	M	F	M	F	M	M	F	M	M	F	F	M	F	F
M	M	M	M	F	F	M	M	M	M	F	F	F	F	F
F	F	M	F	F	F	M	M	M	M	F	M	F	F	F
M	F	M	M	F	M	F	M	F	M	M	F	F	M	M
F	F	F	F	F	M	F	F	F	M	F	M	F	F	M
F	M	M	F	M	F	F	M	M	M	M	F	M	M	F
M	F	M	M	M	M	F	M	F	M	F	F	M	F	F
F	M	F	M	F	M	F	F	M	M	M	M	F	F	F
M	M	F	F	M	M	M	M	M	F	F	F	F	M	M

Let males be represented by odd digits and females by even digits. By use of the 'random number function' and the 'if function' on most good spreadsheets it is quite easy to generate a very large set of five groups of letters where each is either M or F. Such a printout is shown here. By selecting 100 groups and counting those which have two 'M's' it will be seen that the probability is about 30%.

Using Programmable Calculators to Generate a Random Patterns of Letters

Students who possess a programmable calculator can easily generate a random pattern of letters.

Example C: Returning to the bank problem of example B, a student with a programmable calculator such as an *fx*-7700GB can easily generate a group of 5 random letters from the set {M, F} by pressing the [EXE] key five times after entering the following program:

```
Lbl 1 : Int (10 × Ran #) → x : x ≥ 5 ⇒ goto 2 : "M" ◢ goto 1 : Lbl 2 : "F" ◢
goto 1
```

Owners of other brands may need to check their owner's manual in order to write an equivalent program.

Exercise 41b

[Give all answers to one significant figure.]

1. In the bank just mentioned, what is the probability that a group of five will contain exactly one male?

2. People are selected for an interview for a job in the following way. They guess five digits. They will be interviewed if they have two digits which are greater than or equal to 8 and no digit is less than 3. What percentage go forward for an interview?

3. Competitors have to overcome at least five out of ten obstacles. The obstacles are extremely difficult and any competitor has only a 25% chance of overcoming any one of them. What percentage of competitors will fail the competition?

Further Simulations

In the preceding example and exercise we were sampling from populations in which each distinct type in the population was equally distributed. The following example shows how we can create simulations where each type in the population is not equally distributed.

Example D: A factory employs 30% of its workers from a certain ethnic group. Find the probability a group of six workers will contain exactly three of this ethnic group.

O	O	O	E	O	O	O	O	E	O	O	O
O	O	O	O	O	E	O	O	O	O	O	O
O	O	E	O	O	O	E	O	E	E	O	E
O	O	O	O	O	O	O	O	E	E	O	O
O	O	O	E	O	O	O	O	O	O	E	O

O	O	O	O	O	O	O	O	O	E	O	O
O	O	O	O	E	E	O	O	O	O	O	O
E	O	O	O	O	O	E	O	O	O	E	O
O	O	O	O	O	E	E	E	O	O	E	O
O	O	O	O	E	O	E	O	O	E	E	O
O	E	O	E	O	O	O	O	E	O	O	E
O	O	O	O	O	O	O	O	O	E	O	O
O	E	O	E	O	E	E	O	O	O	E	O
O	O	O	E	O	E	O	O	O	O	E	O
E	O	O	E	E	O	O	O	O	O	O	O
O	O	O	O	E	E	O	O	O	E	E	E
O	O	E	O	E	O	O	O	O	O	O	O
E	O	E	O	E	E	E	O	O	E	O	O
E	O	E	E	O	O	O	O	E	O	E	O
O	O	O	E	O	O	E	O	O	E	E	E
O	O	E	O	O	O	O	E	O	O	E	O
O	O	E	E	E	E	O	E	O	E	O	O
E	O	E	E	O	E	O	O	O	O	O	O
E	O	O	O	E	O	E	O	E	O	E	O
O	O	O	O	E	O	O	O	E	O	O	O
O	O	O	O	O	O	E	O	O	O	O	O
O	E	O	E	O	O	O	O	O	O	E	O
O	E	E	E	O	O	O	E	E	E	E	O
O	O	O	O	O	O	O	O	O	O	E	E
E	E	O	E	E	O	O	O	O	O	O	O
E	O	E	E	O	O	O	E	O	O	O	O
O	O	E	O	E	E	O	E	E	O	E	E
E	O	O	O	O	O	O	E	O	E	O	O
E	E	O	O	O	O	E	O	O	O	E	E
O	E	O	O	O	O	O	E	E	O	O	O

Solution: Again, by use of a spreadsheet with 'random number' and 'if' functions a large set of groups of six containing the letters 'E' for 'Ethnic group' and 'O' for 'Others' can be generated in the right proportion. It will be found that about 20% of such groups have exactly 3 from ethnic group E.

Exercise 41c

1. In the same factory as example C what is the probability we have 3 or more ethnic workers in a group of 6?

2. Forty percent of a farmer's sheep are black. Find the probability that if four randomly selected sheep are shorn then two or more will be black.

3. Describe in words how you would try to simulate each of the following using random numbers, dice, etc.
   a. You are required to find out about groups of five people and the numbers of left-handed people in these groups. Assume that 10% of the population is left-handed.

b. You are required to find out about groups of five people and the number of smokers in these groups. It is assumed that 25% of the population smoke.

c. You are required to find out about groups of five people and the number of smokers in these groups, given that 23% of the population is assumed to smoke.

Note: How to create a set of letters, each being B or R, with 80% of the population from which they are sampled being B and 20% R. Using a Macintosh computer and the spreadsheet Excel:

i. Enter 'B' in cell A1.
ii. Enter 'R' in cell A2.
iii. Select a large group of cells below A2, ie. A3 to A30.
iv. Type **= if (100 * rand ( ) > = 20, A$1, A$2)** into the formula bar.
v. Press the [OPTION] and [RETURN] keys. The user will observe that the cells A3 to A30 fill up with 'B' and 'R'.
vi. Select A1 to A30 then also B1 to B30, . . . , J1 to J30.
vii. From the EDIT menu, click on 'FILL RIGHT'.
viii. The worksheet will now be full of 'B's' and 'R's'.
ix. Select print area A3 to J30 and print off your population.

Users of other computers and spreadsheets will have to consult their manuals in order to do this.

Problems and Investigations

1. Consider how you could use a random number generator to investigate a population where there were two groups A and B with numbers in the ratio 6 : 7.

2. Generate randomly fifty digits, each of which must be 0, 1, 2.

3. Investigate how a computer generates random numbers.

4. The game of 'noughts and crosses' is played by players alternately placing O's and X's in empty spaces in a rectangular 3 × 3 grid. The game terminates when one player has a full horizontal, vertical or diagonal line of his or her symbol.

X	O	
X	O	
X		

O	X	
X	O	
		O

Investigate these problems:

a. Suppose a computer alternately places X and O in empty spaces randomly until all nine have an X or O in them. The computer always begins with X.
   i. What is the probability that there is a 'winning line' of X's?
   ii. What is the probability that there is a 'winning line' of X's and a 'winning line' of O's?

b. Suppose the computer stops the game as soon as a 'winning line' is reached. What is the probability that the computer stops before all nine squares are full?

42. THE NORMAL DISTRIBUTION

Achievement Objectives

On completion of this chapter, students should be able, at:

LEVEL 7 STATISTICS

- *to recognise situations where the normal distribution is a suitable mathematical model and use this model to solve problems*
- *to reduce a normal distribution to standard normal form and use tables of normal probabilities*

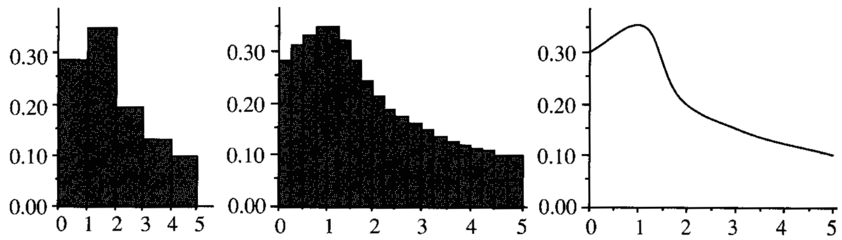
The Probability Density Graph

When a **continuous population** is sampled (refer Chapter 34), the distribution of the sample can be represented on a **relative frequency histogram.**

When larger samples of the population are taken, the width of the intervals on the axis measuring the size of the members of the population becomes smaller. The resulting histogram represents the distribution of the population more accurately.

In the limit, as the size of the sample gets ever larger, the resulting graph is called a **probability density graph.**

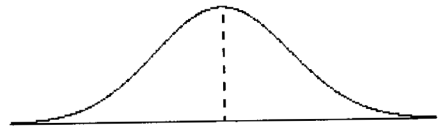
Example A: Consider the following representations of the distribution of a population, where the size of the sample taken gets larger and larger.



The Normal Distribution

In the natural world, many things such as the heights of students in a secondary school, the lengths of worms in a compost bin, or the weights of a species of fish have probability density functions with a bell-shaped appearance. Such phenomena are said to be normally distributed.

A graph of a normally distributed population is symmetrical about the **population mean**, μ, and appears as follows:



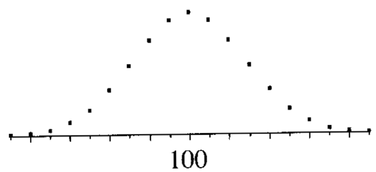
μ

Population Mean

Normal Approximations to a Discrete Population

Strictly speaking, normal distributions are applicable only to continuous data. However, some discrete populations have a normal shaped graph.

Example B: The distribution of measures of human intelligence can be measured by means of an **IQ test**. The IQ value can only take integral values, but its distribution has a normal appearance as shown below:



Calculating Probabilities for a Normally Distributed Variable when the Mean is 0 and the Standard Deviation is 1

Reference is made to normally distributed variables which can be regarded as normally distributed populations.

A variable is a function which generates a set of numbers which have the same type of distribution graph as the population with which it is associated. The variable is a function of the population.

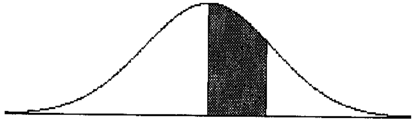
Probabilities for normally distributed variables are calculated using **normal tables** which give the areas under normal probability curves [see appendix]. The areas under normal probability curves measure probabilities.

In the following examples, z (a random variable) has a mean of 0 and standard deviation of 1. The examples show how probabilities correspond to areas under the curves and are calculated using normal tables.

Note: It is wise to sketch the normal distribution for all such problems.

Example C: For the normal distribution shown:

a. the probability that *z* lies between 0 and 0.4 is the area under the curve (shaded) between 0 and 0.4 as shown below:



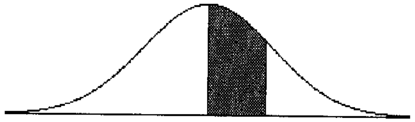
0 0.4

The relevant section of the normal distribution table is shown:

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517	4	8	11	15	19	22	26	30	34
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879	4	7	11	14	18	22	25	29	32
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224	3	7	10	14	17	21	24	27	31

The probability that *z* lies between 0 and 0.4 is found by going across from 0.4 in the *z* column to the column headed 0, giving the value 0.1554, which is the required probability.

b. The probability that *z* lies between 0 and 0.48 is the area shown:



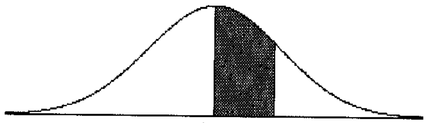
0 0.48

The relevant part of the normal table is shown:

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517	4	8	11	15	19	22	26	30	34
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879	4	7	11	14	18	22	25	29	32
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224	3	7	10	14	17	21	24	27	31

The probability is found by going across from 0.4 in the *z* column to the column headed 8, giving 0.1844, which is the required probability.

c. $P(0 < z < 0.486)$ is the shaded area shown on the normal curve.



0 0.486

The relevant part of the normal table is shown below:

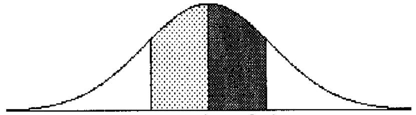
z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517	4	8	11	15	19	22	26	30	34
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879	4	7	11	14	18	22	25	29	32
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224	3	7	10	14	17	21	24	27	31

$P(0 < z < 0.486)$ is found by going across from 0.4 to the column headed 8 giving the value 0.1844. The 6 in 0.486 means that we continue across to the value in the far column headed 6, which is 22. Adding the 22 to the end of the 0.1844 gives $0.1844 + 0.0022 = 0.1866$ as $P(0 < z < 0.486)$.

Although the normal tables only give probabilities for positive *z* values, symmetry allows other values to be calculated.

Example D: Find $P(-0.4 < z < 0)$

Solution: The sketch below shows the required area (shaded lightly) as well as the area (shaded darkly) represented by $P(0 < z < 0.4)$:

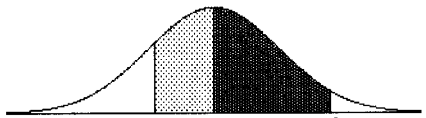


-0.4 0 0.4

By symmetry, $P(-0.4 < z < 0) = P(0 < z < 0.4)$
$= 0.1554$ [from example C, above]

Other probabilities are the sum or difference of two or more areas under the curve.

Example E: The area representing $P(-0.4 < z < 1.2)$ is the area under the curve between -0.4 and 1.2. In the sketch, this area is shown shaded as two separate areas. The darker shaded area is $P(0 < z < 1.2)$ and the lighter shaded area is $P(-0.4 < z < 0)$.

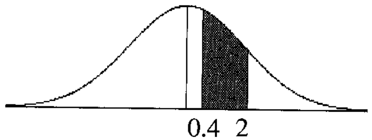


-0.4 0 1.2

Thus $P(-0.4 < z < 1.2) = P(-0.4 < z \le 0) + P(0 < z < 1.2)$
$= P(0 \le z < 0.4) + P(0 < z < 1.2)$
$= 0.1554 + 0.3849$ [from tables]
$= 0.5403$

Example F: Find $P(0.4 < z < 2)$

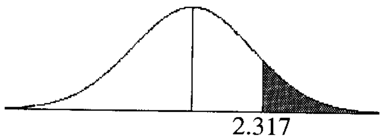
Solution: The required area is shown below:



$$P(0.4 < z < 2) = P(0 < z < 2) - P(0 < z \leq 0.4) \quad \text{[difference of two areas]}$$
$$= 0.4772 - 0.1554$$
$$= 0.3218$$

Normal tables give the areas between the mean 0 and the x value. But some probabilities correspond to the area on the other side of the x value. These are calculated using the fact that the area under the curve is 1, or the area under half the curve is 0.5.

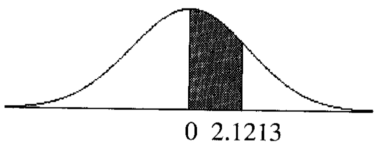
Example G: $P(z \geq 2.317)$ is the area under the curve to the *right* of 2.317:



$$P(z \geq 2.317) = P(z \geq 0) - P(0 < z < 2.317)$$
$$= 0.5 - 0.4898$$
$$= 0.0102$$

Example H: Find $P(0 < z < 2.1213)$

Solution:



Although this probability can be obtained accurately using more advanced methods, we are limited by our tables. The best answer using tables is $P(0 < z < 2.121)$ which is 0.4830.

Exercise 42a: Probabilities when $\mu = 0$ and $\sigma = 1$

z has the standard normal distribution. Use the table from the appendix of this book to find the following probabilities:

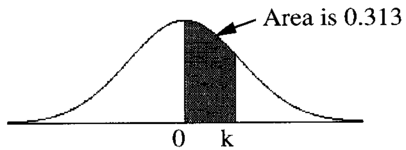
1. $P(0 \leq z \leq 0.5)$
2. $P(0 < z < 1.22)$
3. $P(0 \leq z < 2.6)$
4. $P(1 \leq z \leq 2)$
5. $P(0.2 < z \leq 3)$
6. $P(1.1 < z < 2.45)$
7. $P(0.273 \leq z \leq 1.356)$
8. $P(z \geq 1.2)$
9. $P(z > 1.134)$
10. $P(-0.7 < z \leq 0.1)$

11. $P(-1.12 \leq z \leq 0)$
12. $P(z < -0.982)$
13. $P(z < -2.143)$
14. $P(-0.3 \leq z < 1.2)$
15. $P(-1.723 \leq z \leq 2.3)$
16. $P(-2.32 \leq z \leq -1.1)$
17. $P(z \leq -0.36 \text{ or } z > 1.21)$
18. $P(z < -1.13 \text{ or } z > 1.3)$
19. $P(-0.31 < z < 5.3)$
20. $P(-6 \leq z < 0.3)$
21. $P(z < 0.35)$
22. $P(z \geq -1.3)$
23. $P(z \geq 6)$
24. $P(2z < 3)$

Finding a Value for which a given Percentage are Less Than; Related Problems

Example I: Find k so that the probability that z takes a value between 0 and k is 0.313.

Solution: The following diagram shows the relationship between k and the area beneath the curve:



z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852	3	6	9	12	15	18	21	24	27
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133	3	6	8	11	14	17	19	22	25
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389	3	5	8	10	13	15	18	20	23

By inspecting the normal table, the closest value to 0.313 is 0.3131. This corresponds to beginning in the z column with 0.8, going across to 0.3106 in the 8 column and then to 25 in the far column headed 9 giving 0.3131. Hence the value of k is 0.889.

Exercise 42b:

z is the standard normal distribution. Use the tables in your books to find the following:

1. k such that $P(0 \leq z < k) = 0.1626$
2. k such that $P(0 < z < k) = 0.377$
3. A number L such that $P(0 \leq z < L) = 0.4507$
4. A number L such that $P(0.5 < z < L) = 0.2$
5. $P(0.3 \leq z < k) = 0.14$. Find k.
6. Find k such that $P(-0.2 < z < k) = 0.325$
7. Find k such that $P(z > k) = 0.0562$

8. Find k such that $P(z \ge k) = 0.9123$
9. There is a number k such that $P(z < k) = 0.7$. Find k.
10. k satisfies $P(z < k) = 0.85$. Find k.
11. Find k such that $P(0 < z < k) = P(-0.2 < z < 0.3)$
12. Solve the equation for k. $P(0 < z < 2k) = 0.4$
13. Find k such that $P(z \ge k) = P(1 \le z \le 2)$
14. Solve the equation for k. $P(0 \le z \le 3k + 1) = 0.23$
15. Solve the equation for k. $P(z > 2k - 1) = 0.8321$

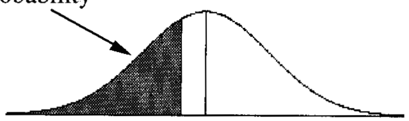
Calculating Probabilities for a Normally Distributed Variable when the Mean is not 0 and the Standard Deviation is not 1

In these situations, problems are treated as shown below and then solved using the same methods as when the mean is 0 and the standard deviation is 1.

If a random variable X has a mean of μ and a standard deviation of σ, then X can be changed into a random variable, Z, which has a mean of 0 and standard deviation 1 using the transformation $Z = \frac{X - \mu}{\sigma}$.

Example J: Fish in a certain lake are found to be normally distributed with a mean weight of 5 kg and a standard deviation of 1.5 kg. Find:
a. the probability that a fish chosen at random will weigh less than 4 kg.
b. the weight that 80% of fish weigh less than.

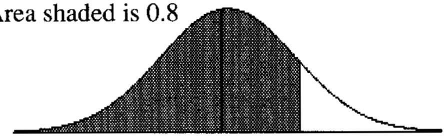
Solution:
a. The probability required is the area shaded:



If X is the weight of a randomly selected fish, then:

$$P(X < 4) = P(Z < \tfrac{4-5}{1.5}) \quad [\text{since } Z = \tfrac{X-\mu}{\sigma}]$$
$$= P(Z < -0.667)$$
$$= P(Z > 0.667) \quad [\text{by symmetry}]$$
$$= P(Z > 0) - P(0 < Z \le 0.667)$$
$$= 0.5 - 0.2477$$
$$= 0.2523$$

b. The area shaded is 80% of the total area under the curve. w is the weight required and is calculated as follows:



Letting the weight be w gives $P(X < w) = 0.8$

$$\therefore P(Z < \tfrac{w-5}{1.5}) = 0.8 \quad [\text{using the transformation } Z = \tfrac{X-\mu}{\sigma}]$$
$$\therefore P(Z \le 0) + P(0 < Z < \tfrac{w-5}{1.5}) = 0.8$$
$$\therefore 0.5 + P(0 < Z < \tfrac{w-5}{1.5}) = 0.8 \quad [\text{by symmetry } P(Z \le 0) = 0.5]$$
$$\therefore P(0 < Z < \tfrac{w-5}{1.5}) = 0.3$$

Inspection of normal tables shows that:

$$\frac{w-5}{1.5} = 0.841$$
$$\therefore w - 5 = 1.2615 \quad [\text{multiplying by 1.5}]$$
$$\therefore w = 6.2615 \text{ kg}$$

Exercise 42c: Probabilities When μ ≠ 0, σ ≠ 1

1. X is a normally distributed variable with mean 2 and standard deviation 4. Find the following:
 a. $P(X > 2)$ b. $P(2 < X < 5)$ c. $P(2 < X < 3)$
 d. $P(3 < X < 5)$ e. $P(X > 3)$ f. $P(X > 1)$
 g. $P(1 < X < 3)$ h. $P(0 < X < 2.5)$ i. $P(-1 \le X \le 3)$
 j. $P(X < -3)$

2. X is a normally distributed variable with mean 3 and standard deviation 2. Find the following:
 a. $P(2 < X < 3.60)$ b. $P(3 < X < 6.20)$ c. $P(4 < X < 6.26)$
 d. $P(X > 5.26)$ e. $P(X > -1.26)$

3. X is a normally distributed variable with mean 5 and standard deviation 3. Find k such that.
 a. $P(X < k) = 0.5$ b. $P(5 < X < k) = 0.2$ c. $P(5 < X < k) = 0.35$
 d. $P(X < k) = 0.65$ e. $P(X < k) = 0.75$ f. $P(X > k) = 0.14$
 g. $P(X \ge k) = 0.22$ h. $P(X \le k) = 0.16$ i. $P(X < k) = 0.45$
 j. $P(X > k) = 0.72$

4. X is a normally distributed variable with a mean of 7 and a standard deviation of 3. Solve for a and b:
 a. $P(X > a) = 0.025$ b. $P(X < a) = 0.95$
 c. $P(a < X < b) = 0.9$ and $P(X < b) = 0.978$ d. $P(X > a) = 0.9$
 e. $P(a < X < b) = 0.8$ and $P(X < a) = 0.1$

5. The young children in a certain region have a mean weight of 40 kg with a standard deviation of 6 kg. If a child is selected randomly:
   a. Find the probability that the child weighs less than 32 kg.
   b. Find the probability that the child weighs more than 27 kg.
   c. Find the probability that the child weighs between 42 and 51 kg.
   d. What is the weight such that the child has a probability of 0.45 of weighing less than that weight?
   e. What is the weight such that the child has a 13% chance of weighing less than that weight?

6. Dogs of a certain species have a mean weight of 18 kg with a standard deviation of 2.5 kg. If a dog is selected at random:
   a. Find the probability that the dog weighs between 17 and 21 kg.
   b. Find the probability that the dog weighs between 13 and 24 kg.
   c. What weight is such that the dog has a 25.78% chance of being less than that weight?

7. A car ride uses a mean of 7 litres of petrol with a standard deviation of 0.5 litres. Assuming that the amount of petrol used is normally distributed:
   a. What is the probability that the car uses more than 7.8 litres on a ride?
   b. What is the probability that it uses between 6.7 and 7.3 litres?
   c. If the car usually has only 8 litres of petrol in its tank, how many times in the next 1 000 rides will the car run out of petrol?

8. A pupil walks daily to high school. The mean time of her walk is 24 minutes with a standard deviation of 3.8 minutes. Assume that the times of her walks to school are normally distributed.
   a. What is the probability that her trip takes at least a quarter of an hour?
   b. If school starts at 8.30 and she leaves at 8.15 daily, what percentage of days would she be late for school?
   c. If she leaves at 8.05 am and her boyfriend gets to school at 8.20 am sharp every day, what percentage of days does she have a chance to see him before class starts?
   d. Find the length of time which 15% of her walks exceed.

Problems and Investigations

Sample Means

If any population (whether normally distributed or not) is sampled by repeatedly taking the same number of elements, the means of the samples will be normally distributed so long as the number of elements in the sample is 30 or more. For some populations, the number in the sample can be considerably less than 30.

In the natural world, many populations are normally distributed because for each phenomenon several factors whose effects are additive are involved. These factors themselves may or may not be normally distributed but as the following practical exercise shows, the effect of averaging a number of values can give sums which are normally distributed. The following exercise should be carried out to confirm the above ideas.

Practical Exercise:

a. Select 120 digits, randomly, from the set {1, 2, 3, 4, 5, 6} and record the selected digits in the following tally chart. [120 rolls of a die would provide the random digits.]

x	Tallies	Frequency
1		
2		
3		
4		
5		
6		

b. Randomly select 120 samples of 6 digits from the set {1, 2, 3, 4, 5, 6}. For each sample of 6 digits find the mean $\bar{x}$ and record the results in the following tally chart.

$\bar{x}$	Tallies	Frequency
$1 \le x \le 2$		
$2 < x \le 3$		
$3 < x \le 4$		
$4 < x \le 5$		
$5 < x \le 6$		

The number of tallies in the first chart will be close to evenly distributed while in the second chart, the sample means will not be.

E.g. If one sample yielded the digits 1, 1, 1, 2, 6 and 5 the mean would be

$$\frac{1+1+1+2+6+5}{6} = 2\frac{2}{3}$$

The result $2\frac{2}{3}$ would give a tally in the $2 < x \le 3$ row.

These ideas can be demonstrated far more dramatically using computers. Samples and sample means of any size can be quickly generated and the relevant graphs drawn. Students with access to a computer should do an extended version of the above exercise.

43. NETWORKS

Achievement Objectives

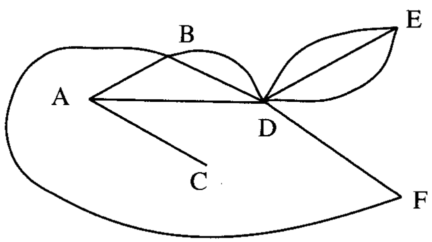
On completion of this chapter, students should be able, at:

- LEVEL 7 GEOMETRY
  - *to choose an appropriate network to organise and visually represent information*
  - *systematically develop, and critically evalute, optimal solutuions using networks*

Networks

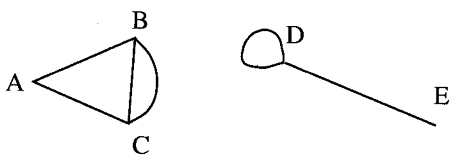
A **network**, often called a graph, is a collection of points called **vertices** or **nodes,** some of which are connected by lines called **edges**. The lines do not have to be straight. Vertices which are joined by an edge are called **adjacent**.

Example A:



This is an example of a **connected network**. Each vertex is connected by edges to at least one other vertex. It is possible to pass between any pair of vertices by travelling along edges.

Example B:



This network is **disconnected** because it is not possible to pass from A to E by travelling along edges. Note the edge that connects D to itself. These are called **loops**.

Exercise 43a:

1. Draw the network containing the vertices A, B, C, D and the edges AB, AC, CD, CD. [**Note:** repetition of an edge indicates that there should be repeated edges joining the points.]

2. Draw the network containing the vertices A, B, C, D, E and the edges AC, AE, BC, BD, BE, BB.

3. Draw the network that connects every vertex in A, B, C, D, E, F with every other with only one edge.

4. Draw a network which shows the following major NZ cities and towns as vertices and edges as connections by major roads:

   Auckland, Whangarei, Hamilton, Tauranga, Taupo, New Plymouth, Wanganui, Palmerston North, Wellington, Nelson, Christchurch, Dunedin.

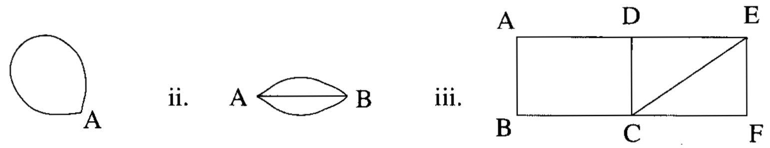
5. Draw a network that features the following people as vertices and edges as relationships either by being brother or sister or by marriage.

Person	Married to	Brother of	Sister of
Tracy	Ben		Mary
Ben	Tracy	Ian	
Mary		Paul	Tracy
Sue	Paul		Shane
Ian		Ben	
Shane		Sue	
Paul	Sue	Mary	

Cycles and Trees

Cycles or circuits in a network are simply defined as paths in the network that begin and end at a common vertex.

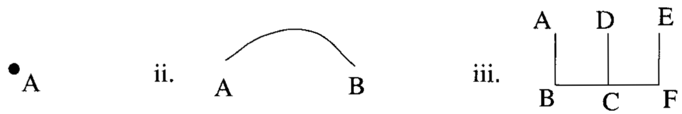
Example C: Each of the following contain at least one cycle.



A **tree** is a connected network that contains no cycles.

A **spanning tree** for a network is a tree which includes every vertex in the network and whose edges are in the network.

Example D: The following are all spanning trees for the networks shown in Example C.

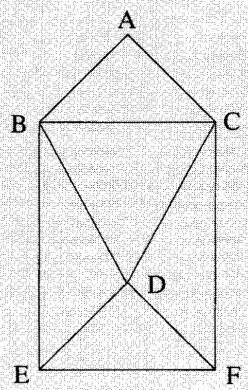


Spanning trees are not unique. There are three possibilities in C ii for example. Can you find them? Can you draw another spanning tree for C iii?

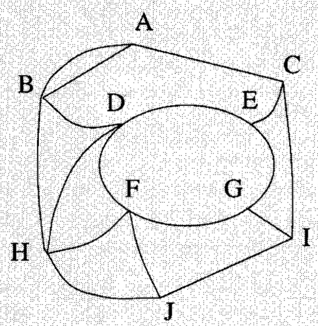
Exercise 43b:

For each of the networks shown draw two spanning trees.

1.



2.



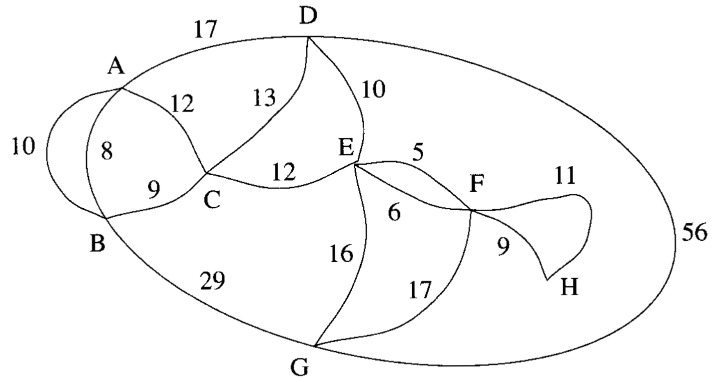
Networks that contain Numerical Information

Most practical applications of networks involve edges which represent numerical values such as monetary value, distance, difference in weight, etc.

Minimal spanning trees

A common problem involving networks is to find what is called a **minimum spanning tree**. This is a spanning tree whose values along the edges have the least possible sum.

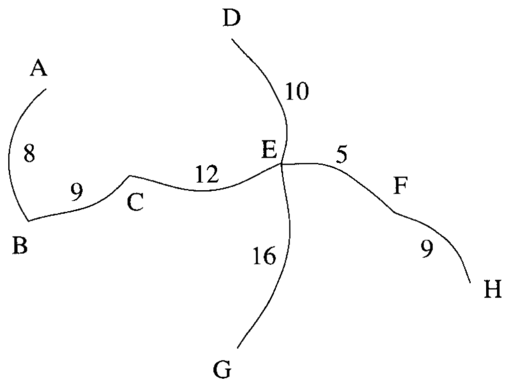
Example E: A rural area has its villages connected by dirt roads. The distances between villages in kilometres are written beside each road represented as an edge. The local government desires to connect all villages by sealed road to keep the total distance sealed to a minimum. The diagram illustrates the region.



Solution:

- start with any vertex, say G.
- connect to the nearest adjacent vertex, in this case E.
- connect EG to the nearest adjacent vertex, in this case F.
- connect GEF to the nearest adjacent vertex which is H.
- connect GEFH to the nearest adjacent vertex which is D

Continuing in this fashion yields the following tree:



Adding the numbers yields a minimum distance of 69km to be sealed.

Interested readers may like to prove for themselves that the network chosen is in fact:

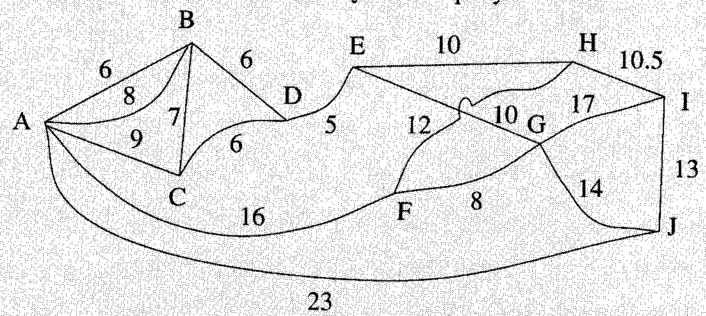
a. a spanning tree.
b. has a total distance which is a minimum.

Exercise 43c

1. A company intends to replace the existing connections between rooms with more modern materials so that the amount of material used is minimal and there is a route between any pair of rooms originally connected.

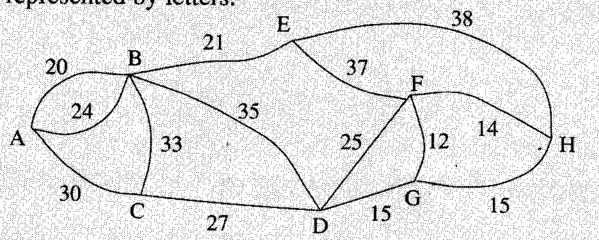
   The diagram shows the rooms and previous connections.

   a. Draw the connections used by the company.



   b. What is the total length of connecting material used?

2. In a historical site there are a number of rooms containing archives and relics of interest. Tourists are confused by the large number of tunnels connecting these rooms. The layout of rooms and tunnels is shown below. Rooms are represented by letters.



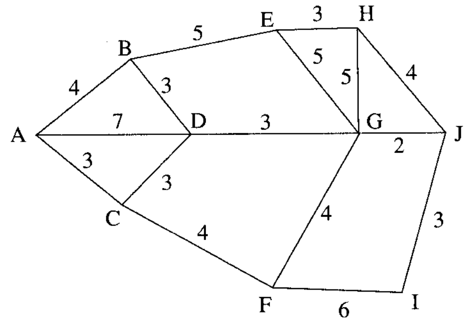
   The numbers represent the lengths of tunnels in metres.

   a. Draw a map showing the tunnels which should be left open so that the total length of tunnel open to the public is a minimum.
   b. Find the total length of the tunnels left open.

Shortest Path

A common problem when dealing with a network is to get from one vertex to another via the shortest possible route.

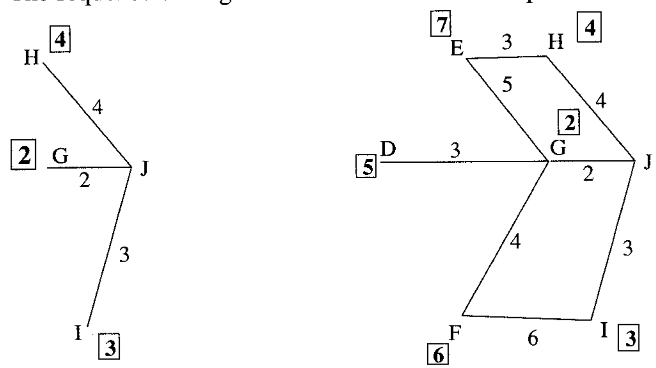
Example F:

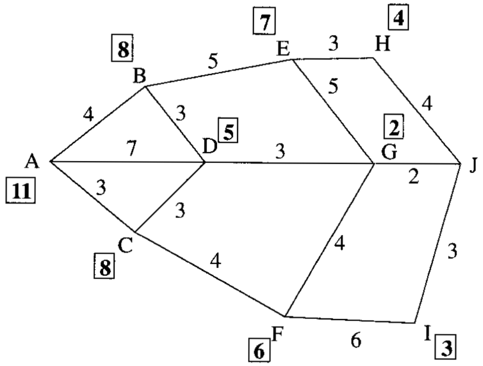


The above network is to be crossed from A to J so that the distance travelled is least.

Solution: This type of problem can be solved by a method called 'dynamic programming' where you start from your finishing point and record the shortest distance possible to any vertex as you proceed.

The sequence of diagrams below illustrate the steps:





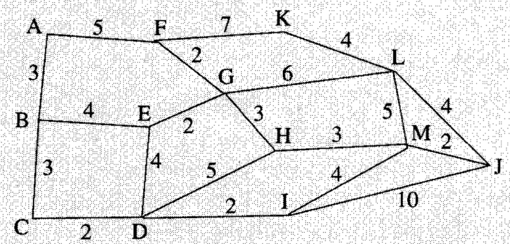
Hence the shortest distance from A to J is 11 and it can be traversed by going along the route A-C-D-G-J.

Exercise 43d

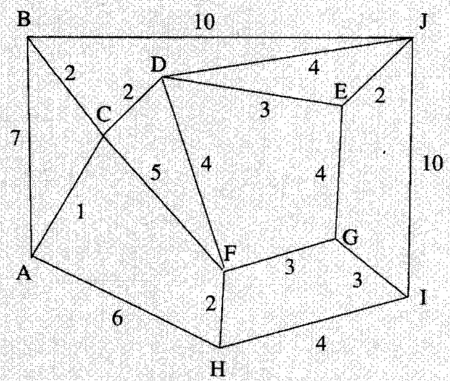
For each of the following diagrams find the shortest path from A to J and indicate the route that should be taken.

[**Note:** Although these diagrams represent edges as straight lines they may not be. In at least one case this must be so.]

1.



2.



Inspection Problems

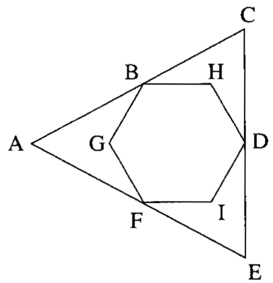
In this type of problem it is necessary to go along every edge in the network but to keep the distance travelled to a minimum.

It is possible to traverse a network going once along each edge under the following conditions:

a. There is an even number of edges at each vertex or;
b. There are exactly two vertices with an odd number of edges. All other vertices have an even number of edges.

Note: Networks of type a. are called Eulerian.
Networks of type b. are called edge-traceable.

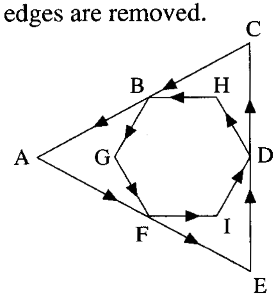
Example G: Find a path through all vertices which only goes along each edge once.



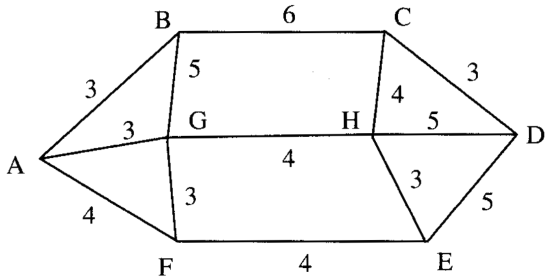
Solution: This network has an even number of edges at every vertex. The method which can be used to find a path which goes through every edge is called Fleury's Algorithm.

i. Select a vertex.
ii. Move along an edge whose removal would not disconnect the network.
iii. Keep moving in this manner until all edges are removed.

This diagram shows one such path with A as the start. The progression being A-F-E-D-C-B-G-F-I-D-H-B-A.



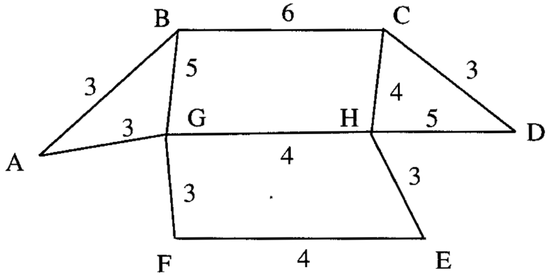
Example H: Find the inspection route of shortest length of all edges in this network.



This network fulfills neither of the criteria for it has six vertices with odd numbers of edges. These are A, B, C, D, E, F. In such situations some retracing is unavoidable. What must be done is to ensure that this is over the shortest possible lengths.

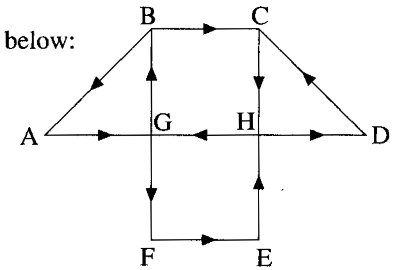
In this diagram the shortest lengths measure 3 hence the route will be planned so that doubling both will occur along such edges.

Removing AF and ED creates this network.



Because there are only two vertices, B and C, with odd numbers of edges we can go along all of the edges starting from B and ending at C.

This route is as shown below:

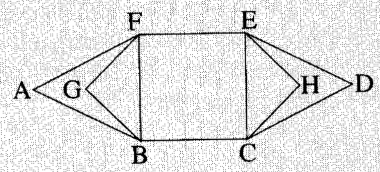


B-A-G-F-E-H-G-B-C-H-D-C

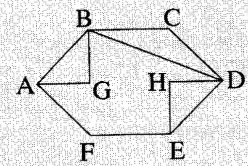
We can now complete the full inspection. This goes F-A-B then the above path then finishes C-D-E. Thus the only edges which are gone over twice are AB and CD both of length 3. Total distance travelled is:
$(3 + 3 + 4 + 5 + 3 + 6 + 4 + 4 + 4 + 3 + 5 + 5 + 3) + (3 + 3) = 58$

Exercise 43e

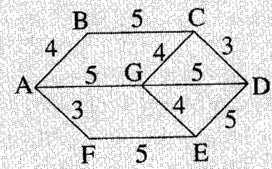
1. Explain why there is a path which goes once along every edge in this network and find it.



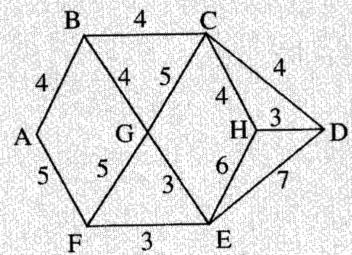
2. Explain why there is a path which goes once through every edge on this network and find it.



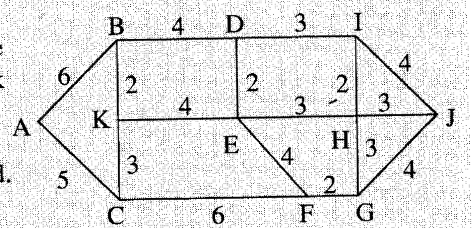
3. a. Find the shortest possible route along the edges of this network allowing passage along every edge.
   b. Calculate the distance travelled.



4. a. Find the shortest possible route along the edges of this network which allows passage along every edge.
   b. Calculate the distance travelled.



5. a. Find the shortest possible route along the edges of this network allowing passage along every edge.
   b. Calculate the distance travelled.

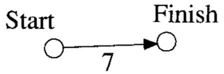


Critical Path Analysis (CPA)

This very important application of network analysis was British in origin and had its beginnings in preparation for World War II.

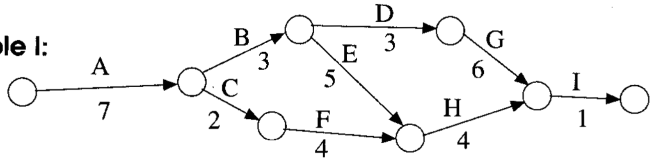
CPA and variations of it have become indispensable tools for people organising large projects.

The first step is to put the tasks which have to be done in correct order. There are a variety of notations used. In this book tasks will be represented by directed lines with circles at their ends. The circle at the left represents 'start'. The circle at the right represents 'finish'. The time taken will be written beneath the line. (Lines do not have to be straight.)



This represents a task which took seven weeks (assuming we are measuring time in weeks).

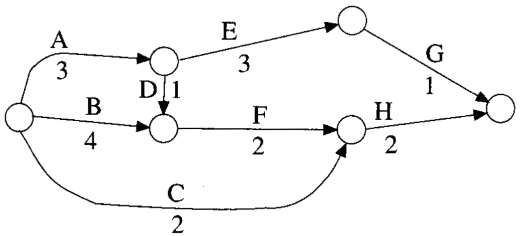
Example I:



This diagram represents a job whose first task was A. B and C depended on A being completed. F depended on C being completed, etc. The dependencies can be represented in a table as below.

Task	Time in weeks to complete	Depends on
A	7	-
B	3	A
C	2	A
D	3	B
E	5	B
F	4	C
G	6	D
H	4	E, F
I	1	G, H

Example J: Describe in words how this project begins and finishes.

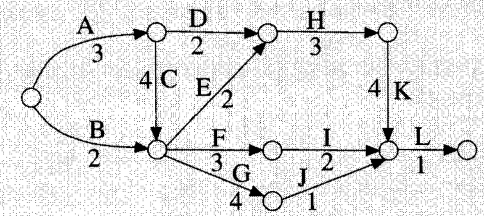


Solution: The project begins with tasks A, B, C. It finished with G, H.

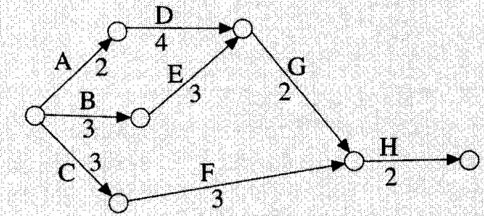
Exercise 43f

1. Write a table of times and dependencies for each of the following networks.

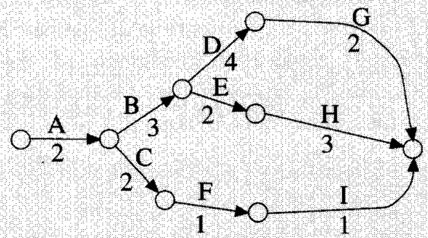
a.



b.



c.



2. Draw networks for the tasks given by these tables.

a.

Task	Time to complete	Depends on
A	2	-
B	3	-
C	4	A
D	4	A
E	2	D, B
F	4	C
G	3	C
H	3	G, E

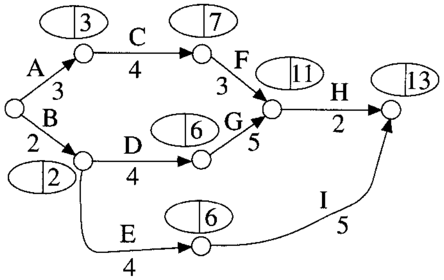
b.

Task	Time to complete	Depends on
A	2	-
B	1	A
C	3	B
D	1	B
E	2	B
F	4	C
G	2	D
H	3	E

Critical Paths

In the performance of a task the **time to completion** is the longest path from start to finish.

Example K: Find a critical path for the diagram below and calculate the time to completion for this network.



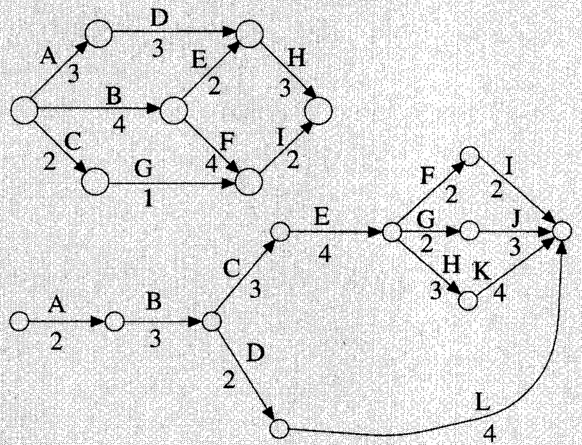
This is calculated by moving from left to right and in the right hand side of the 'split circles' writing the Earliest Finishing Time (**EFT**) or longest time to completion. At the end of F and G we get a completion time of 10 if we go A-C-F and 11 if we go B-D-G. Hence 11 is placed in the right side of the split circle at the end of F and G. Continuing this procedure we find the time to completion is 13 with the critical path being B-D-G-H.

Exercise 43g

For each of the tasks represented by these networks find the time to completion and the critical path.

1.

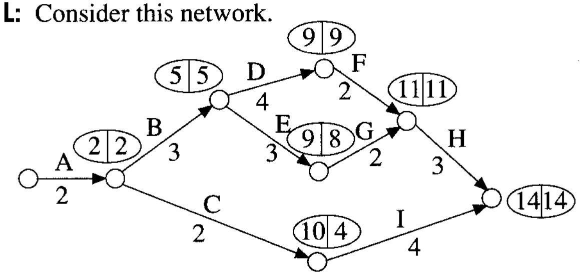
2.



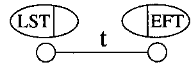
Latest Start Time and Project Management

This final part of critical path analysis examines the Latest Start Times (**LST**) for different tasks of a project. The Latest Start Time is, as the name suggests, the latest time a task can start so that the whole project is on time.

Example L: Consider this network.

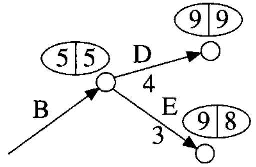


The right sides of the split circles contain the earliest finish times as calculated before. The left sides contain the latest start times. The reader will notice that for LST, t, EFT (as below), t is time required to complete task.



LST = EFT – t

In a case where two edges converge as with D and E in the diagram,



the reader will observe that the LST for the left side of the split circle at the end of B could be calculated using D or E. In a case where the LST would be different depending on which path was taken, the rule is to use the path which gives the smaller LST.

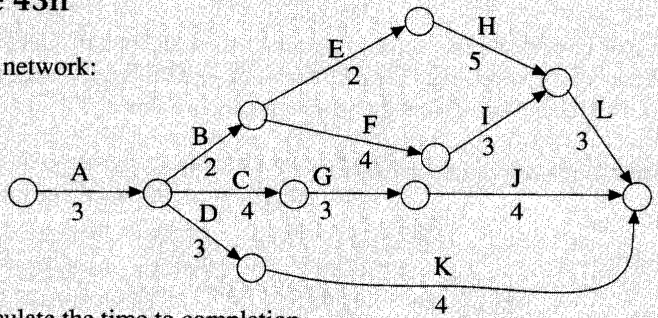
Note: The difference between the LST and EFT is called the **float** for a particular task. It is the amount of 'slouch time' we have between the earliest time it could finish and the latest time it could start. For (10|4) the float is 10 – 4 = 6.

NB: Any point on a critical path has zero float. Here the critical path is A-B-D-F-H.

Warning: If using a network to run any project be sure to update it regularly. The floats and critical paths are never permanent.

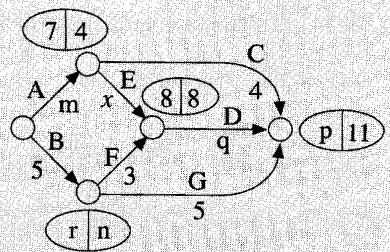
Exercise 43h

1. For this network:



   a. Calculate the time to completion.
   b. Fill in the 'split circles' at every vertex.
   c. Write down the critical path.
   d. What is the largest 'float' at any vertex of this network?

2. For this network, find the values of all lower case letters.



Problems and Investigations

For at least three activities with which you would be familiar set up a task, time to complete, 'depends on' table and hence a network. Use this network to calculate the time to completion for the task.

Hint: Here is a possible table for a project involving organising a trip for people travelling to a holiday destination.

Symbol	Task	Time	Depends on
A	Advertise trip	1 week	-
B	Process replies	1 week	A
C	Book transport	1 day	B
D	Book food	1 day	B
E	Book entertainment	1 day	B
F	Complete preparation	1 week	C, D, E
G	Travel	1 day	F
H	Feed customers	2 hours	F
I	Entertain customers	1/2 day	F

44. POLAR CO-ORDINATES

Achievement Objectives

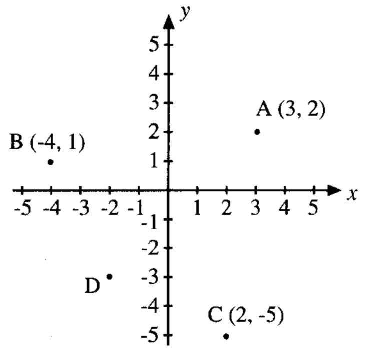
On completion of this chapter, students should be able, at:

- LEVEL 7 GEOMETRY
  - *to convert between rectangular and polar co-ordinates in two dimensions*

Introduction

In their studies of mathematics, most readers will have encountered only the rectangular co-ordinates method of describing the position of a point or object on a plane.

Example A: The diagram shows three points, A, B and C with their positions described by rectangular co-ordinates.



What are the rectangular co-ordinates of the point D?

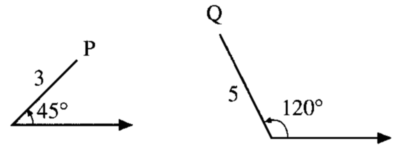
Solution: The answer is (-2, -3).

Polar Co-ordinates

There are other ways of describing a point's position on a plane. One of these is using the method of the **polar co-ordinates**.

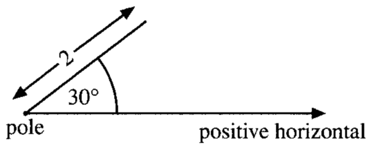
Using this method a point's position is described by its *distance* from a point called the **pole**, and the *angle* that the line joining the point and the pole makes with the positive, horizontal direction, often called the **initial line**.

Example B: In the diagrams the point P has polar co-ordinates (3, 45°), while the point Q has polar co-ordinates (5, 120°).



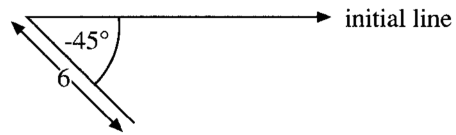
Example C: Plot the point (2, 30°) whose lengths are measured in centimetres.

Solution:
i. Draw the positive horizontal direction and select the pole.
ii. Draw a line making an angle of 30° from the positive horizontal direction.
iii. Mark the point on this line 2 cm from the origin.



Example D: Plot the point (6, -45°).

Solution:



Note: In the polar co-ordinate system the second co-ordinate is an angle θ whose value lies in the set $-180° < \theta \leq 180°$.

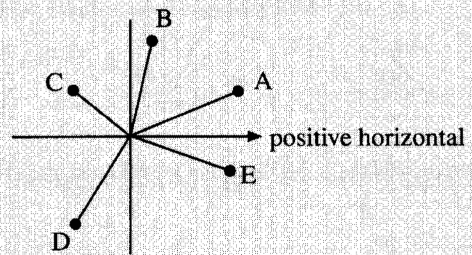
Exercise 44a:

1. Measuring lengths in centimetres and angles in degrees, plot the points having the following polar co-ordinates:
   a. (4, 40°) b. (6, 130°) c. (5, 180°)
   d. (4.5, 220°) e. (6, 315°)

2. Measuring lengths in centimetres and angles in radians, plot the points having the following polar co-ordinates:
   a. (3, 0.37) b. (4, 1.2) c. (5, 2.1)

   [Hint: change the angles to degrees using the relationship R radians = $(\frac{180\,R}{\pi})°$]

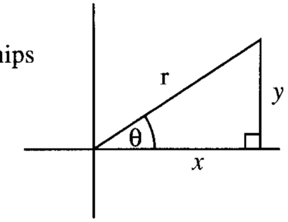
3. Measuring lengths in millimetres and angles in degrees, write down the polar co-ordinates of the points A, B, C, D, E.



4. Draw a pair of x, y axes using centimetres to measure co-ordinates. Plot the following points, then measure the polar co-ordinates of each using a ruler and protractor.
   a. (-2, 4) b. (3, 5) c. (4, -3)
   d. (-4, -5) e. (5, -2)

Converting from one type of Co-ordinate to the other

If the rectangular co-ordinates of a point are (x, y) and the polar co-ordinates are (r, θ) then the relationships can be seen by inspecting the diagram.



x, y, r are the three sides of a right angled triangle, with r being the hypotenuse, x adjacent to θ and y opposite to θ.

$\therefore \frac{x}{r} = \cos\theta$ $\quad [\cos = \frac{\text{adjacent}}{\text{hypotenuse}}]$

$\therefore x = r\cos\theta$

Similarly $y = r\sin\theta$.

Using Pythagoras' theorem we have the relationship:

$r^2 = x^2 + y^2$ [square of the hypotenuse equals the sum of the squares on the other sides]

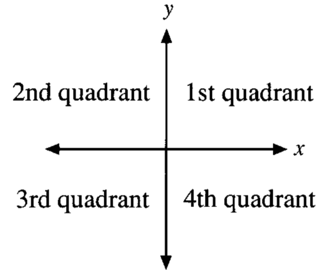
$\therefore r = \sqrt{x^2 + y^2}$ [taking square roots]

$\tan\theta = \frac{y}{x}$ $\quad [\tan = \frac{\text{opposite}}{\text{adjacent}}]$

$\therefore \theta = \tan^{-1}(\frac{y}{x})$ [this result is true in the first quadrant. See summary for points in other quadrants.]

Summary

If the rectangular co-ordinates of a point P are (x, y) and the polar co-ordinates are (r, θ) then:



$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}(\frac{y}{x})$ if P is in the first quadrant.

$\theta = 180° - \tan^{-1}(\left|\frac{y}{x}\right|)$ if P is in the second quadrant.

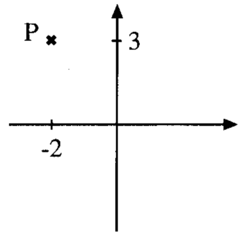
$\theta = -180° + \tan^{-1}(\left|\frac{y}{x}\right|)$ if P is in the third quadrant.

$\theta = -\tan^{-1}(\left|\frac{y}{x}\right|)$ if P is in the fourth quadrant.

$x = r\cos\theta$

$y = r\sin\theta$

Example E: Find the polar co-ordinates of the point P (-2, 3).



Solution: $r = \sqrt{x^2 + y^2}$

$r = \sqrt{(-2)^2 + 3^2}$

$r = \sqrt{13}$

$= 3.6$ (1 d.p.)

P is in the second quadrant, hence:

$\theta = 180° - \tan^{-1}(\left|\frac{y}{x}\right|)$

$= 180° - \tan^{-1}(\frac{3}{2})$

$= 180° - \tan^{-1}(1.5)$

$= 123.7°$

Co-ordinates are (3.6, 123.7°).

Example F: Find the rectangular co-ordinates of the point having polar co-ordinates (4, 30°).

Solution:

$x = 4\cos 30° = 3.5$ (1 d.p.)

$y = 4\sin 30° = 2$

Rectangular co-ordinates are (3.5, 2).

Exercise 44b:

1. Find the rectangular co-ordinates of the points with these polar.co-ordinates:
   a. (5, 35°) b. (7, 43°) c. (5, 115°)
   d. (8, 320°) e. (9, 212°)

2. Find the polar co-ordinates of the points with these rectangular co-ordinates:
   a. (4, 5) b. (-3, 4) c. (-6, -3)
   d. (-2, -5) e. (5, -4)

Using calculators to change from Rectangular Co-ordinates to Polar Co-ordinates and vice-versa

The transformation from one set of co-ordinates to another is very easy using scientific calculators.

Example G: Transform (3, 4) in rectangular co-ordinates into polar co-ordinates.

Solution: The following sequence of keystrokes uses a Casio *fx*-82 calculator. Owners of other models should consult their manuals.

3 SHIFT + 4 = gives r = 5.
SHIFT [(--- gives $\theta = 53.1°$.

(This was carried out in DEGREES mode. If the value for θ had been required in radians then the calculator should have been set to RADIAN mode.)

Example H: Transform (2, -1) in rectangular co-ordinates into polar co-ordinates.

Solution: 2 SHIFT + 1 +/− = gives r = 2.2.
SHIFT [(--- gives $\theta = -26.6°$.

Answer is (2.2, -26.6°).

Example I: Transform (7, 47°) into rectangular co-ordinates.

Solution: Key strokes are:

7 SHIFT − 4 7 = gives $x = 4.8$ (1 d.p.).
SHIFT [(--- gives $y = 5.1$ (1 d.p.)

Exercise 44c:

1. Change the following rectangular co-ordinates into polar co-ordinates:
   a. (5, -2) b. (3, -5) c. (-4, 5)
   d. (5, 2.5) e. (-1.3, 2)

2. Change the following polar co-ordinates into rectangular co-ordinates:
   a. (7, 36°) b. (8.2, 43°) c. (5.4, 117°)
   d. (15, -114°) e. (2.7, -47°)

Problems and Investigations

1. a. A tramper gives his position as 20km from base at a compass bearing of E 30° N. Taking due East as the initial line and the base as the pole, what would be his polar co-ordinates?
   b. Others radio in and their positions are:
      i. 25km from base, compass bearing N 43° E.
      ii. 15km from base, compass bearing W 25° N.
      iii. 5km from base, compass bearing E 52° S.
      iv. 19km from base, compass bearing S 35° W.
      For each, give the polar co-ordinates.

2. Another means of giving compass directions is by using the angle made in a clockwise direction with North. A series of beacons are placed at different positions from headquarters. Taking headquarters as pole and direction East as positive horizontal, find the polar co-ordinates of the following beacons:
   a. 20km in a direction 032°. b. 35km in a direction 119°.
   c. 5km in a direction 212°. d. 17km in a direction 287°.

3. A scientist places recording devices at different positions from her computer.
   a. Device 1 is 105m West and 200m North of the computer.
   b. Device 2 is 250m East and 63m North of the computer.
   c. Device 3 is 315m East and 86m South of the computer.

   For each device find its polar co-ordinates, taking the computer as pole and East as the positive horizontal.

4. Investigate the graphs of the following, given in polar co-ordinates.
   a. $r = 2$ b. $r = 3 \sin \theta$ c. $r = 2 + \sin \theta$
   d. $r = 3 \sin 2\theta$ e. $r = 21 \sin 3\theta$

ANSWERS

Chapter 1: Basic Algebra

Exercise 1a

1.	a. 21	b. 20	c. 48	d. 105	e. 3 150
	f. -12	g. -60	h. 42	i. -12	j. 72
2.	a. 2	b. -10	c. -5	d. -2	e. -19
	f. -11	g. 29	h. -6	i. 16	j. 17
3.	a. 9	b. 16	c. 32	d. 72	e. 12
	f. 26	g. 100	h. 15	i. 26	j. 128
	k. 16	l. 32	m. -125	n. -6	o. 9
4.	a. 2	b. 56	c. 3	d. 45	e. 8
	f. 6	g. 8	h. 1	i. 2	j. $6\frac{3}{4}$
	k. -3	l. 6	m. -27	n. 16	

Exercise 1b

1. a. ? → A : ? → B : ? → C : 4A – B + 2C b. 3
2. a. ? → A : 3AAAA or ? → A : 3A x^y 4 b. 1

Exercise 1c

1.	a. $7x$	b. 2A + 5B	c. 21Q + 5P	d. 10P + 7Q	e. 6P
	f. $5x + 2y$	g. 4A + 9B	h. 3P	i. 9A + 3B	
	j. 18Q + 17B – 9P				
2.	a. 4B	b. AE	c. 20B	d. 12AB	e. 28A
	f. 36CD	g. 20AQ	h. 48AB	i. 15ABP	j. 240CPQR
3.	a. 3A	b. 4B	c. 5	d. 3B	e. 6B
	f. 2B	g. C	h. $\frac{C}{B}$	i. $\frac{B}{2}$	j. $\frac{1}{C}$
	k. $\frac{A}{3C}$	l. $\frac{3}{A}$	m. $\frac{3}{B}$		

Exercise 1d

1.	a. A^2	b. A^3B^2	c. P^2Q^2	d. $2A^2$	e. $12P^2Q$
	f. $42P^2Q$	g. $180P^2Q^2$	h. 270A	i. $3A^2B$	j. $12L^3$
	k. A^3	l. $24A^3$	m. P^4Q^2	n. $12M^4$	o. AB^2C^2
2.	a. 8A + 2B	b. -3P	c. 8AB + 3BC	d. 8A + 2AB	e. $-3xy$
	f. -5A – 2B	g. $7A^2 + 7A$	h. 4AB	i. 3PQ – 3P	j. $17 - 6x$
	k. 10P – 11	l. 11A + 3B + 2C		m. A + 2B – 3	n. 7AB – 7PQ
	o. $10A^2 - 7$				
3.	a. A	b. $3x$	c. 5A	d. $\frac{7PA}{Q}$	e. $10A^2$
	f. A	g. A	h. M		i. 40MN
	j. 2P	k. 64P	l. $5M^2$	m. 2B	
4.	a. 15AB	b. $6A^3$	c. 24ABP	d. A^5	e. $20P^{14}$
	f. $36P^2Q$	g. $9P^3$	h. $48R^3$	i. $72A^3P^2$	j. $2A^2B^2CD$
	k. 5	l. 4B	m. $4x$	n. $\frac{4x}{y}$	o. x^3
	p. A	q. 2A	r. $12B^2$	s. $\frac{1}{2A}$	t. $\frac{1}{2A}$

Exercise 1e

1. $18x$	2. $72x^2$	3. $3x^6 + 4x$	4. $10x^8$	5. $12x^8$
6. $3x^{23}$	7. $19y$	8. $18x^5y^2$	9. $2y^6 + 3y$	10. $30y^9$
11. $48y^9$	12. $\frac{y^3}{2}$	13. 20a	14. $16a^5$	15. $3a^8 + 2a^2$
16. $3a^{10}$	17. $24a^{10}$	18. $3a^{18}$	19. 14b	20. $48b^3$
21. $5b^4$	22. $512b^{12}$	23. $72b^{12}$	24. b^{19}	25. $3x^4$
26. $3x^3$	27. $\frac{13x}{12y}$	28. $4x^2 + 5xy$	29. $6x^2 - 13x$	30. $2xy$
31. $7x^2$	32. $13x$	33. $-\frac{2}{5x}$	34. $60x^3$	35. $-10x^2$
36. $25y^2$	37. $5y^2$	38. $16y^2 - 3y$	39. $-\frac{3}{4}$	40. $36y^5$
41. $48y^3$	42. $-12y^3$	43. $-36y^5$	44. $2x^6 + 3x^2 - x^4$	
45. $40x^2$	46. x^4	47. $18x^3$	48. $9x^4$	49. $4x^5$
50. $25x^2$				

Chapter 1 – Problems and Investigations

2. $P = \frac{1}{x+y+z}$ The claim is true. 3. True when B = D and A + 2B = 2C.

4. i. $2x^{n+1}$ ii. $\frac{2x^2}{3}$ iii. $\frac{2x^5}{5}$ iv. $\frac{9}{4x^2}$ v. $\frac{x^3}{2^4 3^{m-2}}$

Chapter 2: Linear Equations

Exercise 2a

1.	a. $2\frac{1}{3}$	b. $2\frac{1}{11}$	c. $\frac{3}{8}$	d. -1	e. 5.3	f. 3.6
	g. -2.73	h. $\frac{2}{3}$	i. 8	j. 0.55		
2.	a. 2	b. -6.5	c. -5.5	d. 7	e. -4	
	f. 6	g. 27	h. -6	i. 2.5	j. $6\frac{1}{6}$	
3.	a. 12	b. $-1\frac{1}{4}$	c. $-1\frac{3}{7}$	d. $7\frac{1}{5}$	e. $-11\frac{1}{4}$	f. $\frac{16}{21}$
	g. $\frac{35}{4}$	h. $\frac{4}{5}$	i. -12.8	j. $3\frac{2}{3}$		
4.	a. $1\frac{3}{4}$	b. $1\frac{1}{3}$	c. $\frac{7}{3}$	d. $7\frac{3}{4}$	e. $6\frac{1}{6}$	f. $2\frac{2}{7}$
	g. $1\frac{3}{7}$	h. $2\frac{1}{7}$	i. -7	j. 15		
5.	a. 4	b. $-6\frac{1}{2}$	c. $8\frac{1}{2}$	d. 10	e. 2.4	f. $2\frac{4}{7}$
	g. $-7\frac{3}{4}$	h. $14\frac{1}{3}$	i. $-20\frac{1}{2}$	j. -32		

Exercise 2b

1.	a. $\frac{1}{8}$	b. $2\frac{5}{8}$	c. $1\frac{1}{19}$	d. $\frac{1}{4}$	e. $9\frac{1}{4}$	f. $-1\frac{3}{8}$
	g. $\frac{8}{11}$	h. $2\frac{1}{3}$	i. 1.6	j. $-\frac{2}{3}$	k. $-12\frac{3}{4}$	l. $\frac{1}{7}$
2.	a. $3\frac{3}{7}$	b. $\frac{2}{5}$	c. 3	d. 1	e. $2\frac{3}{4}$	f. $-2\frac{1}{2}$
	g. -42	h. $-1\frac{6}{7}$	i. $1\frac{12}{19}$	j. 1	k. $1\frac{3}{4}$	l. 4
	m. 2					

Exercise 2d

1. $\frac{2x+4}{5} - \frac{(x-2)}{8} = 3$ 2. 1 3. 7 and 8
4. = (2 * A68 + 4) / 5 – (A68 – 2) / 8 5. 1.9444 (4 d.p.)

Exercise 2e

1. (A + 7) ÷ 2 – (A – 3) ÷ 4 replaces (A + 5) ÷ 5 – (A – 2) ÷ 12
2. More repetitions of the program would be necessary. No changes otherwise.
3. 0.153 or 0.154

Exercise 2f

1. \$2.40 2. Tom \$1.50; Bruce, Dave \$2.50; rest have \$3.00 each
3. \$35.50, \$15.50 4. 14.15 m 5. 26 6. 18 7. \$7
8. 55 km 9. 15 10. \$7 11. $13\frac{1}{2}$% 12. 143 13. 40
14. \$8; \$32 15. \$3500

Exercise 2g

1. $x > 1\frac{2}{3}$ 2. $x \le 2\frac{1}{2}$ 3. $x \le 3$ 4. $x > -2$ 5. $x \ge -5$ 6. $x < -17$
7. $x \le -43$ 8. $a \ge 3\frac{2}{11}$ 9. $x > 1\frac{4}{5}$ 10. $x \ge -\frac{11}{25}$
11. a. 167.5 A + 45 < 1070 b. 6 12. a. $1.3x + 20 > 108$ b. 68
13. a. 17.2 – 0.21a < 0.8 b. 79 14. a. $\frac{50 - Da}{3} + \frac{2(60 - De)}{5} > 15$ b. \$31.40
15. a. $\frac{C-3}{4} + \frac{G-7}{6} < 15$ b. 52

Chapter 2 – Problems and Investigations

1. \$660 2. 11 cars, 26 bikes, 18 tricycles

Chapter 3: Expanding and Factorising

Exercise 3a

1. $15x + 5$	2. $10x^2 + 15x$
3. $2x^2 + 3xy + y^2$	4. $6x^2 + 11x + 4$
5. $12A^2 + 7A - 10$	6. $2x^2 + x - 21$
7. 12PQ + 18P – 16Q – 24	8. $6A^2 + AB - B^2$
9. $3x^2 + y^2 - 4xy + 2x - 2y$	10. $3x^2 + 4y^2 + 8xy - 3x - 6y$
11. $2x^2 + y^2 - 3xy + 8x - 6y + 8$	12. $x^2 + 2x + 1$

13. $9x^2 - 24xy + 16y^2$
14. $P^2 + R^2 - 4P - 4R + 4 + 2PR$
15. $A^3 + 3A^2B + 3AB^2 + B^3$
16. $x^4 - 8x^2 + 16$
17. 8
18. $12A^4 - A^3B + 13A^2B^2 + 7AB^3 + 5B^4$
19. $x^4 - 4x^3 + 10x^2 - 12x + 9$
20. $8x^3 + 36x^2y + 54xy^2 + 27y^3$

Exercise 3b

1. $7x + 11$
2. $7A + B$
3. $6x + 3y$
4. $8x + 8y + 14$
5. $12x + 19y$
6. $3A - 17B$
7. $x^2 + 5x - 2$
8. $2x^2 - 12$
9. $2x^2 + 6x + 5$
10. $2x + 7$
11. $-8x^4 - 12x^3 + 17x^2 + 17x - 11$
12. $6A^3 + 16A^2 - 1$
13. $48A^2 + 48A + 28$
14. $4x^2 - 3x + 1$

Exercise 3c

1. $a(x - b)$
2. $a(b - a)$
3. $x(11x - 1)$
4. $4a(x + 2y)$
5. $(x + 4)(x + 3)$
6. $(x + 9)(x + 4)$
7. $(P + 3)(P + 9)$
8. $(P - 2Q)(P - Q)$
9. $(R - 2P)(R - 4P)$
10. $(x - 7y)(x + 6y)$
11. $(P + 5Q)(P - Q)$
12. $(P - 5Q)(P + 2Q)$
13. $(R + 6S)(R + 4S)$
14. $(x - 7)(x + 7)$
15. $P(P - 36)$
16. $(L - 3)(L - 2)$
17. $(M + 10)(M + 6)$
18. $(M - 8)(M + 1)$
19. $25(M - 2)(M + 2)$
20. $(2x + 1)(x + 1)$
21. $(x - 5b)(x - b)$
22. $(2x + 1)(x + 3)$
23. $(4P + 1)(P + 5)$
24. $(3z + 7)(z - 1)$
25. $(4P - 7)(P + 3)$
26. $(4x - 1)(3x - 2)$
27. $(LP - 6)(LP + 4)$
28. $(7 - 5x)(2 + 3x)$
29. $(7p - q)(p + q)$
30. $8(L - 2x)(L + 2x)$
31. $3P(3x - y)(3x + y)$
32. $2(5 - x)(5 + x)$
33. $A(3P - 1)(P - 1)$
34. $x^2(9 + x^5)$
35. $x(18y - 21x - 14)$
36. $8p^3q^3(1 + 2p + 3p^2q)$
37. $(p^3q^2r^2 - 3x^5y^6)(p^3q^2r^2 + 3x^5y^6)$
38. $(x - b)(x + b)(x^3 + b^3)$
39. $(A - B)(C - D)$
40. $(x + 1)(x + 1 + a)$

Exercise 3d

1. $(x + 1)(x^2 - x + 1)$
2. $(z - 2)(z^2 + 2z + 4)$
3. $(3 - z)(z^2 + 3z + 9)$
4. $(4x - 3k)(16x^2 + 12xk + 9k^2)$
5. $(m - n)(m + n)(m^4 + m^2n^2 + n^4)$
6. $(A - B)(x + y)$
7. $(m + n)(nm - n^2 + m + n)$
8. $(x - 1)(x + 5)$
9. $(x + y - a - b)(x + y + a + b)$
10. $(bz - a)(az - b)$

Exercise 3e

1. $\frac{x-5}{x+1}$
2. $\frac{x+2}{x+3}$
3. $\frac{x}{x+1}$
4. $\frac{x+7}{x+2}$
5. $\frac{1}{x+1}$
6. $\frac{x+5}{x-4}$
7. $\frac{p+7}{p+3}$
8. $\frac{5}{p+1}$
9. $\frac{a-b}{b}$
10. $\frac{y-3}{4y}$
11. $\frac{2y+1}{y+1}$
12. $\frac{a-b+c}{b-a+c}$
13. $\frac{a^2-a+1}{a+1}$
14. $\frac{x+y}{z}$
15. $\frac{x+b}{x+a}$

Chapter 3 – Problems and Investigations

1. b.ii. 4
   d. $n^3 + (n + 1)^3$
   $= ((n + 1)^3 + n^3 + (n - 1)^3 + \ldots + 3^3 + 2^3 + 1) - ((n - 1)^3 + (n - 2)^3 + \ldots + 3^3 + 2^3 + 1^3)$
   $= \frac{1}{4}(n + 2)^2 (n + 1)^2 - \frac{1}{4}n^2 (n - 1)^2$
2. b. $\frac{1}{4}n^2 (n + 1)^2$ c. 45

Chapter 4: Rational Expressions

Exercise 4a

1. a. $\frac{3a}{4b}$ b. $\frac{5ac}{bd}$ c. $\frac{3ac}{4db}$ d. $\frac{a^2}{bc}$ e. $\frac{b}{2}$ f. $3p$
   g. $15a$ h. $\frac{a}{6}$ i. $\frac{2}{3}k^2$ j. $\frac{13b}{8}$ k. $\frac{y^3}{x^3}$ l. $\frac{y}{3x}$
2. a. $\frac{x}{x+1}$ b. $\frac{1}{x-1}$ c. $\frac{x-2}{x+2}$ d. $\frac{4}{x+4}$

Exercise 4b

1. a. $\frac{2c}{3a}$ b. $\frac{abe}{cd}$ c. $\frac{3x^2}{y^2}$ d. $\frac{a}{5}$ e. $\frac{b}{c}$ f. $\frac{3b}{a}$
   g. $2a$
2. a. $\frac{2a+2}{15}$ b. $\frac{5(x^2-1)}{7x}$ c. $\frac{4b-4}{9}$ d. $\frac{2x+2y}{3y}$ e. $\frac{9x-3}{4}$ f. $\frac{6x}{5x+5}$
3. a. 5 b. $\frac{x+2}{x+1}$ c. $(x + 1)(x - 2)$ d. $\frac{x+3}{x+1}$ e. $\frac{x-2}{x+5}$

Exercise 4c

1. a. $\frac{3a+b^2}{3b}$ b. $\frac{3a}{8}$ c. $\frac{a+1}{b}$ d. $\frac{4a+b}{4b}$ e. $\frac{4a-3b}{12}$ f. $\frac{a^2-bc}{ac}$
   g. $\frac{a+2}{a^2}$ h. $\frac{2a+3}{3b}$ i. $\frac{10+b}{4}$ j. $\frac{21p+2q}{7}$
2. a. $\frac{2a+3}{2}$ b. $\frac{3b+4a}{3}$ c. $\frac{7a+17b}{6}$ d. $\frac{4a+4}{3}$ e. $\frac{15b+31}{6}$ f. $\frac{9a+2}{2a}$
3. a. $\frac{a+7}{12}$ b. $\frac{13b+6}{15}$ c. $\frac{-p-7q}{12}$ d. $\frac{11p+8q}{20}$ e. $\frac{14a-12}{15}$ f. $\frac{21-5p}{6}$
   g. $\frac{26+5p}{10}$ h. $\frac{1+7r}{12}$ i. $\frac{6p-4q}{3}$
4. a. $\frac{2x+3}{(x+1)(x+2)}$ b. $\frac{3p+5}{(p+3)(p+1)}$ c. $\frac{3p-5}{(p+1)(p-3)}$ d. $\frac{5R+2}{(R+4)(R-2)}$
   e. $\frac{2R+9}{R(R+3)}$ f. $\frac{R-A-1}{(A+3)(R+2)}$ g. $\frac{3A-1}{(A-3)(A-1)}$ h. $\frac{2A^2+2A-2}{(A+1)(A-1)}$
   i. $\frac{3x-2}{6(x+1)}$ j. $\frac{xy-3x^2-4y^2}{12(x+y)(x-y)}$ k. $\frac{5x+9}{(x+1)(x+2)(x+3)}$ l. $\frac{x+8}{x(x+1)(x+2)}$
   m. $\frac{3x-y+1}{(x-y+2)(x+y-3)}$ n. $\frac{x-5y+10}{x(x+y-2)(x-y+2)}$ o. $\frac{4y-3x}{xy(x-y)}$
5. a. $\frac{a}{3b^3}$ b. $\frac{4y^2z^4}{9x^2}$ c. $\frac{3a(a+c)}{c^2}$ d. $\frac{x+2}{x(x-3)}$ e. $\frac{(u-6)(x-1)}{(x+4)(u-1)}$
   f. $\frac{4(a+2)}{3(a-2)}$ g. $\frac{5}{2x-2y}$ h. $\frac{5x+1}{x^2-1}$ i. $\frac{x+6y}{x^2-4y^2}$ j. $\frac{x}{x^2-y^2}$
   k. $\frac{2x}{(x+1)(x+2)}$ l. $\frac{2}{(a-4)(a-5)(a-6)}$
   m. $\frac{4x+2}{(x-1)(x+1)(x+3)}$ n. $\frac{(a-1)(a-2)(7a-12)}{12}$

Chapter 4 – Problems and Investigations

2. d. $\frac{A}{BC} = \frac{A}{B(C-B)} - \frac{A}{C(C-B)}$

Chapter 5: Formulae

Exercise 5a

1. 100D ¢ 2. a. $\frac{x}{20}$ hours b. $180x$ seconds
3. a. $(1000x + y)$ grams b. $(x + \frac{y}{1000})$ kg
4. a. 100LW cm^2 b. $\frac{LW}{100}$ m^2 c. (2000L + 20W) mm
5. a. 800L cm b. 8000L mm 6. a. $\frac{1}{2}ab$ b. $\sqrt{a^2+b^2}$
7. a. $\frac{\pi L^3}{4}$ b. $\frac{3\pi L^2}{2}$ 8. a. $2y + 2z$ b. $yz - xy + x^2$
9. a. $2(L + M) + 8x$ b. LMx 10. a. $16L^3$ b. $40L^2$ 11. 5L
12. $\frac{1}{2}L^2$, where L is the length of the rectangle. 13. 3L
14. $(\pi + 2)R$ where R is the radius. 15. $\frac{H^2}{2}(H-1)$ 16. $\frac{4}{3}\pi R^3$
17. a. $\pi R^2 H$ b. 18
18. a. $2(B + L + 1)$ b. No effect provided b is less than $\frac{1}{2}B$
19. a. $\frac{L^2}{4}$ b. $\frac{5L^3}{2}$ c. $\frac{31L^2}{2} + 5\sqrt{5}\,L^2$
20. a. $\pi(R^2 - r^2)$ b. $2\pi(R + r) + R - r$
21. a. $\sqrt{42h}$ b. $\sqrt{42}\,(\sqrt{1.8} - \sqrt{1.5}\,)$ km, (approximately 0.76 km)
22. a. $\frac{N}{2}(S + L)$, where S is smallest and L is largest, and N is the number of numbers.
 b. 3725
23. $\frac{169}{512}\pi R^2 H$ 24. a. $2(L + \pi R)$ b. $R(\pi R + 2L)$ c. 4R+2L
25. a. $\$0.12x$ b. $\$\frac{3.6xz}{y}$

Exercise 5b

1. $\frac{T}{3}$ 2. $\frac{T}{A}$ 3. PT 4. P^2 5. AP + BP
6. $\frac{5P}{2}$ 7. $\frac{T^2}{5}$ 8. A – B 9. A + B 10. 2B
11. $2B^2 + 3B$ 12. B^2 13. AB^2C 14. B(C + 3) 15. $\frac{AC}{B}$
16. $\pm\sqrt{A}$ 17. $\pm\sqrt{\frac{A+B}{C}}$ 18. $\frac{B^2+D}{A}$ 19. $\frac{B^2}{A-D}$ 20. $\frac{ABC}{A+B}$
21. $\frac{ED+A}{1-E}$ 22. C^2 23. C^2D^2 24. $\frac{9L^2G}{4A^2}$ 25. $\frac{A^2G}{C^2X^2}$
26. $\sqrt{\frac{D}{3-A}}$ 27. $\frac{D^2}{(A-B)^2}$ 28. $\frac{(A+C)^2}{E^2}$ 29. $\frac{4y-5}{By-A}$ 30. $\sqrt[3]{\frac{Fy-E}{4y-3}}$

Exercise 5c

1.a. 45.08 m^2 b. 71.13 m^2 c. 3.44 m d. 1.94 m e. 10.82m
2.a. 817.24 cm^3 b. 106.55 cm^3 c. 3.44 cm d. 5.70 cm e. 5.33 cm

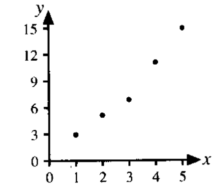
Chapter 5 – Problems and Investigations

1. Numbers are 3, 6. Product is 108. 2. 9 orders 3. value is 9

Chapter 6: Graphs I

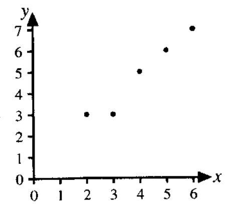
Exercise 6a

1. a. {1, 2, 3, 4, 5,}
 b. {3, 5, 7, 11, 15}
 c.



 d. No repetition of first elements.
 e. Graph of T^{-1} consists of the points (3, 1), (5, 2), (7, 3), (11, 4), (15, 5) only.
 f. Yes. No repetition of first elements.

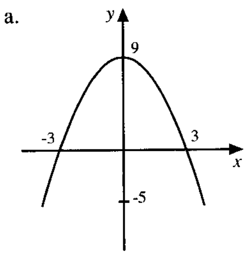
2. a. {2, 3, 4, 5, 6}
 b. {3, 5, 6, 7}
 c. Graph consists of only the 5 points.



 d. A has no repetition of first elements; A^{-1} has.

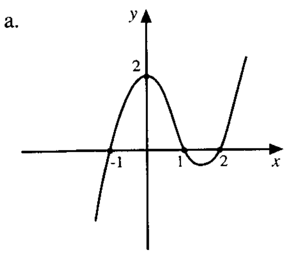
Exercise 6b

1. a.



 b. $x = \pm 3$ c. $y = 9$

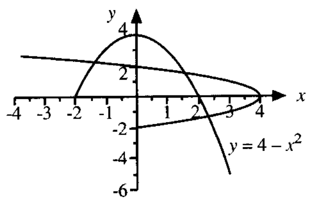
2. a.



 b. $x = -1, 1, 2$ c. $y = 2$

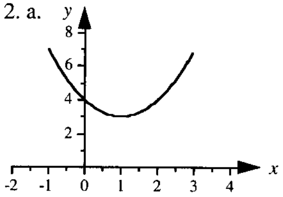
Exercise 6c

1. a.



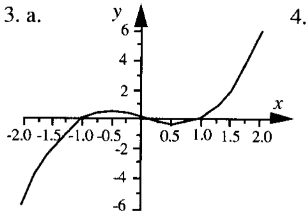
 b. $x = \pm 2$ c. $y = 4$
 d. $-2 \le x \le 3$ e. $-5 \le y \le 4$
 f. $0 < x \le 3$

2. a.



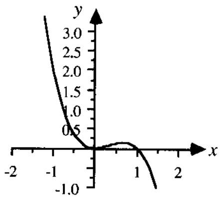
b. (1, 3); minimum
c. $1 < x \leq 3$
d. (0, 4)
e. Symmetrical about vertical line $x = 1$

3. a.



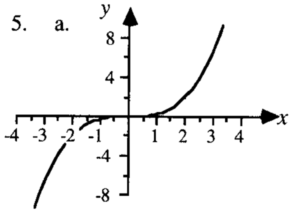
b. (0.6, -0.4); minimum
c. (-0.6, 0.4); maximum
d. Point symmetry about (0, 0).
e. $x = -1, 0, 1$
f. $y = 0$
g. Domain: $-2 \leq x \leq 2$
Range: $-6 \leq y \leq 6$

4. a.



b. $x = 0, 1 \quad y = 0$
c. (0, 0) minimum
$\left(\frac{2}{3}, \frac{4}{27}\right)$ maximum
d. $-2 \leq x < 0; \frac{2}{3} < x \leq 2$
e. No
f. Domain: $-2 \leq x \leq 2$
Range: $-4 \leq y \leq 12$

5. a.



b. Point symmetry about (0, 0)
c. $f(x) \to \infty$ as $x \to \infty$, $f(x) \to -\infty$ as $x \to -\infty$
d. $x = \pm 2, 0$
e. Domain and range are R

6. Domain: $-4 \leq x \leq 4$ — Range: $0 \leq y \leq 4$
x intercept: $x = -4$ — y intercept: $y = 2$
Turning points: Nil — Symmetries: Point symmetry about (0, 2)
As $x \to \infty$: N.A. — As $x \to -\infty$: N.A.
Infinity: Nil — Discontinuity: Nil

7. Domain: $-3 \leq x \leq 4$ — Range: $-4 \leq y < 0, 2 \leq y \leq 4$
x intercept: Nil — y intercept: $y = 2$
Turning points: Nil — Symmetries: Nil
As $x \to \infty$: N.A. — As $x \to -\infty$: N.A.
Infinity: Nil — Discontinuity: $x = 0$

8. Domain: $x \geq -5$ — Range: $0 \leq y \leq 2, y = 4$
x intercept: $x = -5$ — y intercept: $y = 2$
Turning points: Nil — Symmetries: Nil
As $x \to \infty$: $y \to 4$ — As $x \to -\infty$: N.A.
Infinity: Nil — Discontinuity: $x = 0$

9. Domain: $x \geq -4$ — Range: R
x intercept: $x = -4$ — y intercept: $y = 2$
Turning points: Nil — Symmetries: Nil
As $x \to \infty$: $y \to \infty$ — As $x \to -\infty$: N.A.
Infinity: $x = 0$ — Discontinuity: $x = 0$

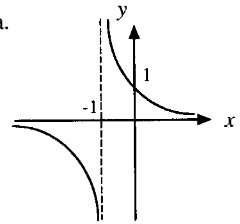
10. Domain: R — Range: $-3 \leq y \leq 4$
x intercept: $x = \pm 1$ — y intercept: $y = 4$
Turning points: (-2, -3), (2, -3), (0, 4) — Symmetries: y axis is axis of symmetry
As $x \to \infty$: $y \to 0$ — As $x \to -\infty$: $y \to 0$
Infinity: Nil — Discontinuity: Nil

11. Domain: $x \geq -2$ — Range: $y \geq 0$
x intercept: $x = -2$ — y intercept: $y = 2$
Turning points: Nil — Symmetries: Nil
As $x \to \infty$: $y \to 0$ — As $x \to -\infty$: N.A.
Infinity: $x = 0$ — Discontinuity: $x = 0$

12. Domain: $-4 < x < 0, 0 < x \leq 3$ — Range: $y \geq 2$
x intercept: Nil — y intercept: Nil
Turning points: Nil — Symmetries: Nil
As $x \to \infty$: N.A. — As $x \to -\infty$: N.A.
Infinity: $x = 0$ — Discontinuity: Nil

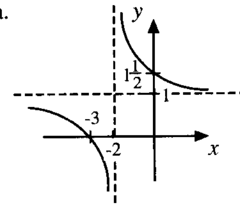
Exercise 6d

1. a.



b. No
c. x intercept does not exist, $y = 1$
d. $g(x) \to -\infty$ as $x \to -1$ from below
$g(x) \to +\infty$ as $x \to -1$ from above
e. Point symmetry about (-1, 0)
f. Domain $\{x: x \neq -1\}$
Range $\{y: y \neq 0\}$
g. Horizontal asymptote: y = 0
Vertical asymptote: $x = -1$
h. No points of inflection. No points on graph where 'wiggling' occurs

2. a.

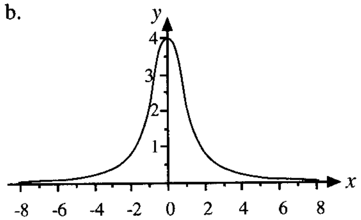


b. Domain $\{x: x \neq -2\}$
Range $\{y: y \neq 1\}$
c. $y \to 1$ as $x \to +\infty$, and as $x \to -\infty$
d. $x = -2$
e. $y \to \infty$ as $x \to -2$ from above
$y \to -\infty$ as $x \to -2$ from below
f. Horizontal asymptote: $y = 1$
Vertical asymptote: $x = -2$
g. No points of inflection. No points on graph where 'wiggling' occurs

3. a.

x	-100	-5	-3	-1	0	1	3	5	100
y	0.0004	0.15	0.4	2	4	2	0.4	0.15	0.0004

b.



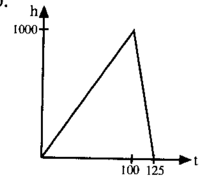
c. Domain R; Range $\{y: 0 , y \leq 4\}$
d. $y \to 0$ as $x \to \pm\infty$
e. $\{x: x < 0\}$
f. horizontal asymptote: $y = 0$
g. 2

Exercise 6e

1. a.

Time (minutes)	0	25	50	75	100	115	125
Height above base of hill (m)	0	250	500	750	1000	400	0

b.



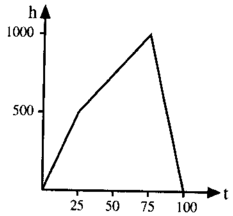
c. i. 300m ii. 800m iii. 520m
d. 80 minutes and 105 minutes
e. i. Rising during the first 100 minutes
ii. Descending during the last 25 minutes
f. Domain is $0 \le t \le 125$; t is in minutes
Range is $0 \le h \le 1000$; h is in metres

2. a. 75 minutes

b.

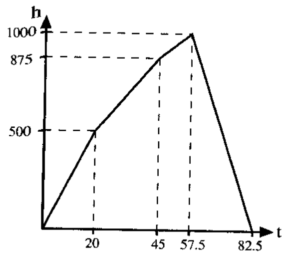
Time	0	10	25	50	75	90	100
Height	0	200	500	750	1000	400	0

c.

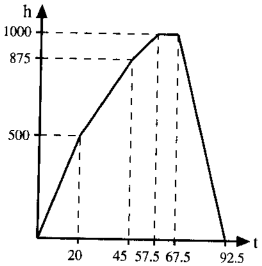


d. i. 550 ii. 800 iii. 0
e. 55 minutes, 80 minutes
f. i. Rising during the first 75 minutes
ii. Descending during the last 25 minutes
g. Domain is $0 \le t \le 100$, t is in minutes
Range is $0 \le h \le 1000$, h is in metres

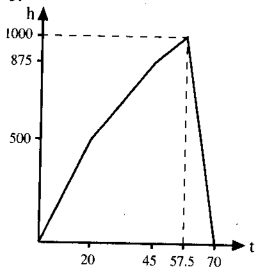
3. a.



b.



c.



4. a. Steady climb at 25 m a minute for 40 minutes. Steady descent at $16\frac{2}{3}$ m a minute.
b. Steady climb at $8\frac{1}{3}$ m a minute for 60 minutes. Then climb at increased rate of 25 m a minute for 20 minutes. Rest for twenty minutes. Descent at a rate of $16\frac{2}{3}$ m a minute.
c. Climb at rate of 25 m a minute for 20 minutes. Then goes back at a steady rate of 12.5 m a minute. Then climbs at rate of 25 m a minute, then goes back at a steady rate of about 12.5 m a minute. Climb finishes at a steady rate of $8\frac{1}{3}$ m a minute thereafter. Information was no longer gathered at the end of this climb.

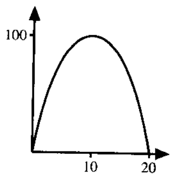
Chapter 6 – Problems and Investigations

1. a. i. 15 cm ii. 75 cm²

b.

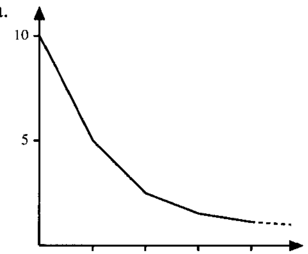
Length	1	2	4	6	8	10	12	14	16	18
Area	19	36	64	84	96	100	96	84	64	36

c.



d. $0 < l < 20$, $0 < A < 100$
e. 100

2. a.



b. 9 minutes

Chapter 7: Simultaneous Equations

Exercise 7a

1. $x = 3, y = 2$
2. $x = 0, y = 3$
3. U = 1, V = 5
4. R = -2, T = 2
5. $A = \frac{7}{2}, B = 2$
6. $x = \frac{17}{23}, y = \frac{36}{23}$
7. $x = 3, y = 6$
8. $x = 13, y = -4$
9. $x = -2, y = -3$
10. $x = 8, y = 42$
11. $x = 2\frac{1}{2}, y = 3$
12. $x = -\frac{4}{3}, y = 4$
13. $y = -0.61, x = 23.34$
14. $x = 7, y = 2$
15. Infinite Solutions
16. $x = \frac{2}{11}, y = \frac{1}{2}$
17. $(\pm 2, \pm 1)$
18. $x = \frac{1}{2}, y = 1$
19. $x = 4, y = \pm\sqrt{3}$
20. $x = \frac{1}{9}, y = 4$

Exercise 7b

1. 63, 75
2. Maths 1hr, English = $1\frac{1}{2}$ hrs
3. (A) $1.04/kg (B) 52c/kg
4. Peter $600, Michael $980
5. AC = 12, Boris = 23
6. Mary 10 km, Sue 8 km
7. 25 men and 71 women
8. $A = -\frac{1}{5}, B = \frac{2}{5}$
9. A = 3, B = -1
10. A = 0, B = +6, other factor $(x - 2)$

Exercise 7c

1. (0, -1), (1, 0)
2. (0, 1), $(-\frac{4}{5}, -\frac{3}{5})$
3. (0, 2), $(-\frac{4}{3}, \frac{2}{3})$
4. (0, -3), $(\frac{12}{5}, \frac{9}{5})$
5. (0, 3), $(-\frac{36}{13}, -\frac{15}{13})$
6. (-2, -1), (1, 2)
7. (5, 2), (-2, -5)
8. (2, -2), (72, 12)
9. (4, 3), $(\frac{9}{4}, \frac{16}{3})$
10. $(\frac{3}{2}, 4)$, (2, 3)
11. (-3, -2), (2, 3)
12. (2, 1), (1, 2)
13. (4, 2), (-2, -4)
14. $(\frac{16}{3}, 4)$, $(\frac{4}{3}, -2)$
15. $(1, \frac{1}{2})$, $(-\frac{2}{3}, 3)$

Chapter 7 – Problems and Investigations

2. 17 novels, 23 others
3. $x = 6$, a = 4
4. $x = 17, y = 15$
5. $x = 10, y = 10, z = 23$

Chapter 8: Solution of Polynomial Equations

Exercise 8a

1. a, b, c, f, h, i, j

2. a. 3 b. 5 c. 6 d. 4 e. 1 f. 2
   g. 3 h. 4 i. 2 j. 0
3. a, c, e, f, are quadratics; b, g, are cubics

Exercise 8b

1. a. i. 4 ii. 18 iii. -2 iv. 0 v. $1\frac{3}{4}$
   b. $x = -2, 1$ c. $x^2 + x - 2$
2. a. -3 b. 30 c. $-1\frac{1}{9}$ d. $-4\frac{4}{9}$ e. 33.48
3. a. i. 6 ii. -4 iii. $-2\frac{1}{4}$ iv. $8\frac{4}{9}$ v. $-6\frac{8}{81}$
   b. i. 12 ii. $8\frac{3}{4}$ iii. 56 c. i. 22 ii. 12 iii. $-1\frac{1}{3}$
   d. i. -4, 1 ii. -4, -3 iii. $-\frac{2}{3}$, 1
4. a. i. 0 ii. $-\frac{9}{8}$ ii. 8 iv. 5.247 v. -2.288
   b. i. 1, 2, 4 ii. 1, 2, -1 iii. -1, 2, -3 iv. 1, 2, $1\frac{1}{2}$ v. 1, 2 vi. 2, $\frac{1}{9}$
   vii. -2, 1, 2, 4 viii. 1, 2, 8

Exercise 8c

1. -1, 4 2. -5, 2 3. -2, 4 4. -4, -9 5. -3, 6 6. ±4
7. -3, 4 8. -8, 1 9. -2, 0 10. -2, 6 11. -5, 1 12. $-1, \frac{1}{3}$
13. $-1\frac{1}{2}, -\frac{1}{2}$ 14. ±2 15. ±2

Exercise 8d

1. a. -3.73, -0.27 b. -7.61, -0.39 c. 3.35, 0.15 d. -2.83, 0.83 e. -2.27, 1.77
   f. 0.23, -1.33 g. 1.81, -0.29 h. -8.24, 0.24 i. 1.55, -0.22 j. -4.44, -0.06
2. a. -5.83, -0.17 b. 0.18, 2.82 c. 2.29, -0.29 d. -2.30, 1.30 e. -2.26, 0.59
   f. 0.70, 4.30 g. -1.87, 5.87 h. -0.58, 0.58 i. -0.83, 4.83 j. 1

Exercise 8e

1. width is 9 m 5 cm, length is 11 cm 5 cm 2. width is 10 m, length is 20 m
3. 3.58 m 4. 7.83 m 5. 24, 53 6. 61, 62 7. 2.25 m
8. 9.44 9. 0.73 m, 1.73 m, 2.73 m 10. 4.79 or 0.21
11. a. 31 b. 45 12. height = 3.28 m, base = 2.28 m 13. $8.50

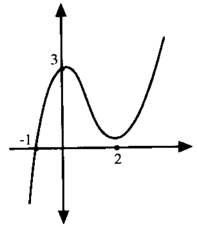
Exercise 8f

1. a. i. 1 ii. 2 b. i. -7 ii. 0 c. i. -15 ii. 0
   d. i. 0 ii. 1 e. i. -20 ii. 0 f. i. -23 ii. 0
   g. i. 85 ii. 2 h. i. -60 ii. 0 i. i. 0 ii. 1
   j. i. 40 ii. 2
2. a. $\frac{1}{4}$ b. $-5\frac{1}{3}$ c. $\pm\frac{4\sqrt{2}}{3}$ d. $k < \frac{25}{24}$ e. $p > \frac{9}{4}$

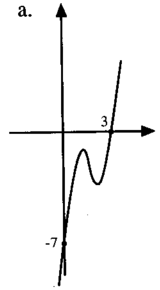
Exercise 8g

1. 2 2. -2.1 3. b. -0.2

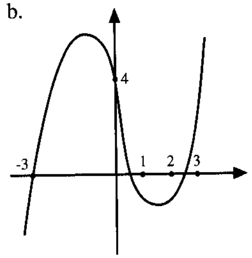
4. a. $x = -1$
   b.



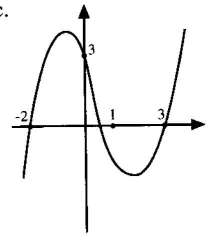
5. a.



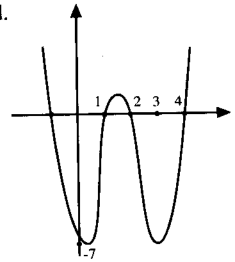
b.



c.



d.



6. a. $y = x^3 + 3x^2 - 6x - 10$, is one possibility
   b. $y = x^3 - 6x^2 + 3x + 11$, is one possibility

Chapter 8 – Problems and Investigations

Max volume = 882 cm³
Max surface area = 576 cm²

Chapter 9: Co-ordinate Geometry of the Straight Line

Exercise 9a

1. a. $\sqrt{32}$ b. 10 c. 13 d. $\sqrt{40}$ e. $\sqrt{41}$
2. a. $\sqrt{5}$ b. $\sqrt{37}$ c. $\sqrt{13}$ d. $\sqrt{74}$ e. $\sqrt{34}$
3. $AB = \sqrt{26}$, $AC = \sqrt{26}$, $BC = \sqrt{32}$ 4. Side lengths are $\sqrt{40}$, $\sqrt{40}$, $\sqrt{32}$
5. (-1,2)
6. a. Sides are $\sqrt{20}$, $\sqrt{5}$, 5 b. Sides are $\sqrt{52}$, $\sqrt{13}$, $\sqrt{65}$
   c. Sides are $\sqrt{26}$, $\sqrt{104}$, $\sqrt{130}$ 7. a. $x = 2$ or -4 b. $x = -7$ or 17
8. $y = 5$ 9. $x = 1$ or -4
10. $AB = \sqrt{53}$, $CD = \sqrt{37}$, $BC = \sqrt{37}$, $DA = \sqrt{53}$
    $\text{Area} = \frac{1}{2}AC.BD = \frac{1}{2}\sqrt{128}\,.\sqrt{50} = \frac{8\sqrt{2}\,.\,5\sqrt{2}}{2} = 40$
11. 24.166 km 12. 72.691 km

Exercise 9b

1. a. $m = \frac{1}{3}$ b. $m = \frac{3}{4}$ c. $m = 1$ d. $m = -2$
   e. m is undefined f. $m = 0$

2. Both lines have gradient 2
3. Both lines have gradient 3
4. Gradients are 1 and 3
5. Not parallel
6. Parallel
7. k = 3
8. k = 4
9. k = -3
10. k = 11
11. $x = 1$
12. $x = 13$
13. $x = -13$
14. 4
15. -4

Exercise 9c

Solutions must be straight lines which go through

1. (0, 1), (5, 3)
2. (-2, 4), (2, 7)
3. (3, 2), (7, 3)
4. (4, 3), (7, 1)
5. (2, 1), (4, 0)
6. (-2, -2), (1, -2)
7. (-3, 0), (-3, 2)

Exercise 9d

1. a. 21.8º b. 36.9º c. 14.0º d. 146.3º e. 153.4º f. 0º g. 90º
2. a. 18.4º b. 36.9º c. 45º d. 116.6º e. 90º f. 0º
3. a. $\frac{3}{2}$ b. $\frac{7}{4}$ c. $\frac{4}{3}$ d. $\frac{9}{20}$ e. $-\frac{13}{20}$

Chapter 9 – Problems and Investigations

1. Collinearity
2. True
3. True
4. True

Chapter 10: Equation of the Straight Line

Exercise 10a

1. The straight lines should go through the indicated points:
   a. (0, 1), (1, 3) b. (0, -4), (1, -1) c. (0, 1), (4, 2) d. (0, 2), (3, 4)
   e. (0, 5), (1, 4) f. (0, 3), (4, 0) g. (0, 3), (5, 1) h. (0, 2), (1, 2)
   i. (-3, 0), (-3, 1) j. (0, -5), (1, -5)
2. a. 2, 63.4º b. 3, 71.6º c. $\frac{1}{4}$, 14.0º d. $\frac{2}{3}$, 33.7º e. -1,135º
   f. $-\frac{3}{4}$, 143.1º g. $-\frac{2}{5}$, 158.2º h. 0, 0º i. Undefined, 90º j. 0, 0º
3. a. 5 b. -9 c. 3 d. 1 e. -4
   f. -3 g. $3\frac{1}{6}$ h. $\frac{2}{3}$ i. -1.5 j. $-7\frac{1}{2}$
4. a. $x = 2, y = -4$ b. $x = -2, y = 6$ c. $x = 6, y = -4$ d. $x = 2, y = 2$ e. $x = 7\frac{1}{2}, y = 5$

Exercise 10b

1. $y = x + 1$
2. $y = 2x + 3$
3. $y = 3x - 2$
4. $y = 4x + 3$
5. $y = \frac{1}{2}x - 4$
6. $y = \frac{2}{3}x + 2$
7. $y = -2x + 3$
8. $y = -\frac{3}{4}x + 2$

Exercise 10c

1. $y = x - 1$
2. $y = 2x + 3$
3. $y = \frac{3}{4}x + 2\frac{1}{2}$
4. $y = -x + 4$
5. $y = \frac{2}{3}x - 4$
6. $y = 4$
7. $y = -\frac{1}{2}x$
8. $y = -\frac{3}{4}x + 2$

Exercise 10d

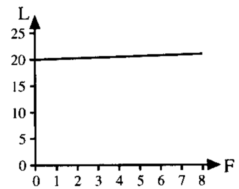
1. c, d, h
2. a. $y = \frac{2}{3}x + \frac{5}{3}$ b. $y = \frac{3}{2}x - \frac{7}{2}$ c. $y = \frac{3}{4}x - \frac{5}{4}$ d. $y = \frac{3}{2}x - 2$ e. $y = \frac{4}{3}x + \frac{19}{18}$
3. b. ii is a straight line equation

Exercise 10e

1. a.

Force	0	1	2	3	4
Length	20	20.125	20.25	20.375	20.5

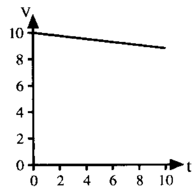
b.



c. 60.8 N

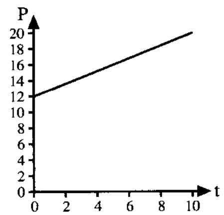
L = 20 + 0.125F

2. a.



b. V = 10 - 0.12t c. After 40 minutes

3. a.



b. P = 12 + 0.8t c. 60 pages

4. a. V = 20 000 - 0.25d b. 20 000km
5. a. 0.1 L per minute b. 3.8 litres c. 562 minutes or 9 hours 22 minutes
6. a. D = 85t + 15.83 b. 15.83 km

Chapter 10 – Problems and Investigations

1. 2nd, 4th days
2. Dec 30, Dec 26, Dec 23, Dec 21, Dec 17

Chapter 11: Parallel and Perpendicular Lines

Exercise 11A

1. $y = 2x - 2$
2. $y = \frac{1}{2}x + 4\frac{1}{2}$
3. $y = -3x + 7$
4. $y = -\frac{2}{3}x + 6$

5. $y = -\frac{1}{3}x + 6$ 6. $y = -\frac{1}{2}x + 7$ 7. $y = 2x - 5$ 8. $y = -\frac{3}{2}x + 3$
9. $2x - 3y - 7 = 0$ 10. $y = 7x + 7$ 11. $2y + 3x - 2 = 0$ 12. $3x - 4y - 15 = 0$
13. $5x - 2y + 10 = 0$ 14. $7x + 4y + 7 = 0$ 15. a. $k = \frac{8}{3}$ b. $k = -\frac{3}{2}$

Exercise 11b

1. a. (3.5, 6) b. (-2, 2) c. (4, -3) d. (1.5, 0) e. (0.5, 4)
2. $y = 2x - 2$ 3. $y = x + 2$ 4. $y = \frac{x}{2} + 0.5$ 5. $y = \frac{x}{2} + 4.5$ 6. $y = x + 4$
7. $y = 2x - 1$ 8. $y = \frac{x}{3} + 3$ 9. $y = -\frac{2}{3}x + \frac{8}{3}$

Chapter 11 – Problems and Investigations

1. $y = -\frac{x}{2} + 11.5$ 2. (4, 2) 3. Area ADE is larger.

Chapter 12: Intersection of Two Straight Lines

Exercise 12

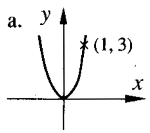
1. a. (1, 2) b. (0, 3) c. (2, 3) d. (2, 1) e. (-2, 0)
   f. (1, 1) g. (2, 1) h. (3, 1) i. (3, 1) j. $(\frac{33}{14}, -\frac{1}{28})$
2. a. (1, 4) b. (1, 5) c. (2, 2)
3. $\frac{5}{2}$ 5. Parallel lines $m = \frac{2}{3}$

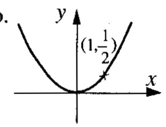
Chapter 12 – Problems and Investigations

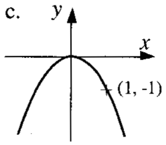
1. 40 weeks 2. 16 weeks 3. 22.62 4. 3 5. The claim is true

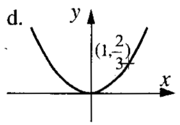
Chapter 13: Graphs II

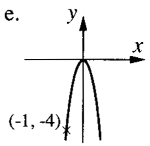
Exercise 13a

1. a.


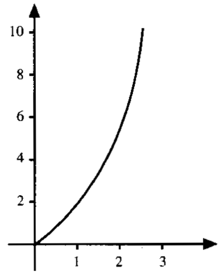
b.


c.


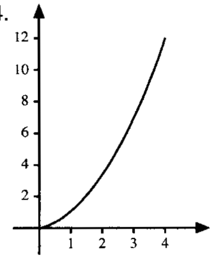
d.


e.


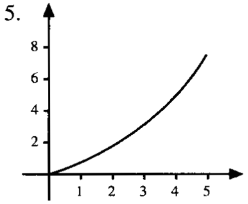
2. a. $y = 4x^2$ b. $y = \frac{1}{4}x^2$ c. $y = \frac{3}{4}x^2$ d. $y = -\frac{1}{3}x^2$

3.


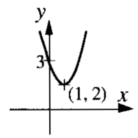
Equation is $A = \frac{1}{2}\pi R^2$

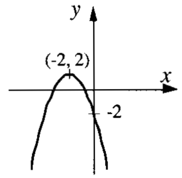
4.


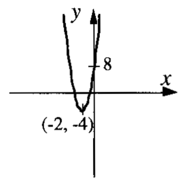
Equation is $A = \frac{1}{4}\pi D^2$

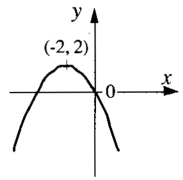
5.


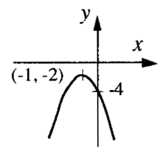
Equation is $P = 0.3L^2$

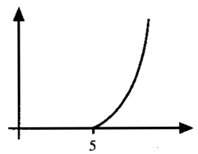
6. a.


b.


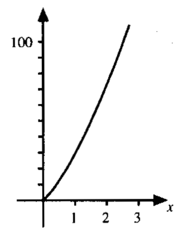
c.


d.


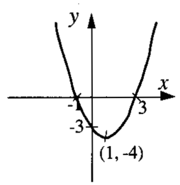
e.


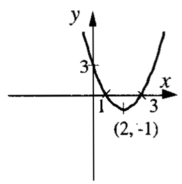
7. a. $y = (x-2)^2$ b. $y = 5 - x^2$ c. $y = (x-3)^2 + 2$ d. $y = (x+4)^2 + 1$
   e. $y = 7 - (x+3)^2$ f. $y = 5 - (x-2)^2$ g. $y = (x-3)^2$ h. $y = (x-2)^2 + 3$
8. a. $y = 2(x-1)^2 + 2$ b. $y = 3(x+3)^2 - 2$ c. $y = 5 - \frac{1}{2}(x-2)^2$
9.


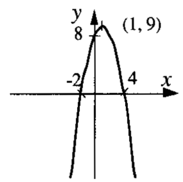
Equation is $A = L^2 - 25$

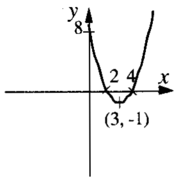
10.


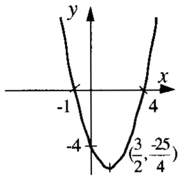
Equation is $A = (2x + 8)^2 - 64$, where x is distance from fence

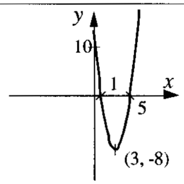
11. a.


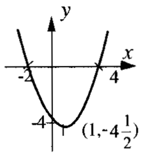
b.


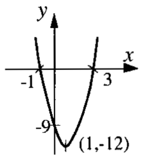
c.


d.


e.


f.


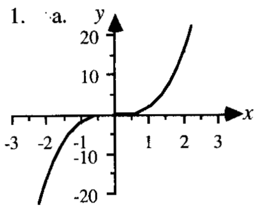
g.


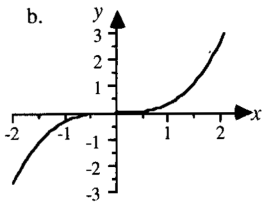
h.


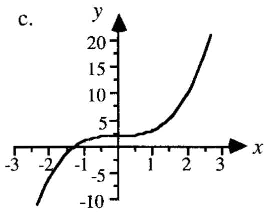
12. a. i. $k = 2$ ii. $(3, -2)$
b. $B = \frac{1}{4}$
c. $L = 4$

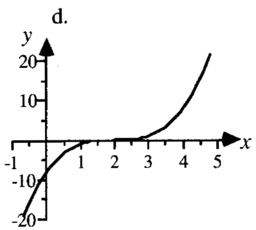
13. a. $y = (x+3)(x-1)$ b. $y = \frac{2}{3}(x-1)(x-6)$ c. $y = 5 - 2(x-1)^2$ d. $y = 2 + 3(x+1)^2$

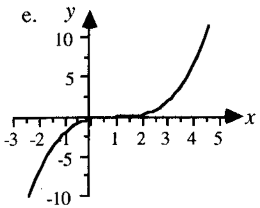
Exercise 13b

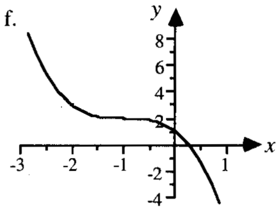
1. a.


b.


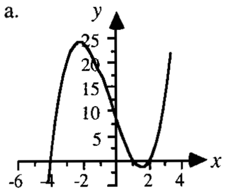
c.


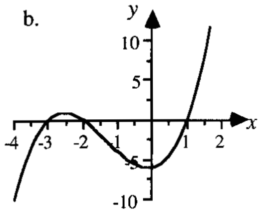
d.


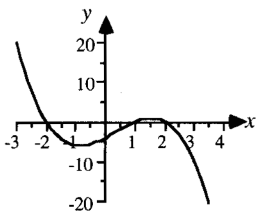
e.


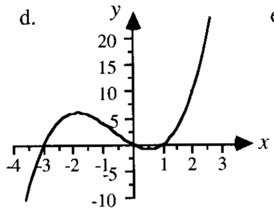
f.


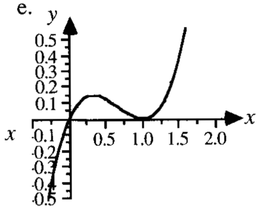
2. a $y = \frac{x^3}{8}$ b. $y = \frac{x^3}{8} + 1$ c. $y = (x+2)^3 + 1$ d. $y = -\frac{1}{2}(x-2)^3$
e. $y = (x-2)^3 + 2$

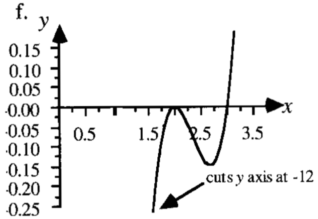
3. a.


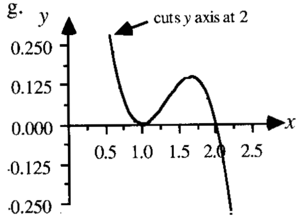
b.


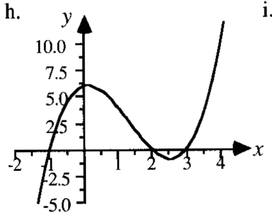
c.


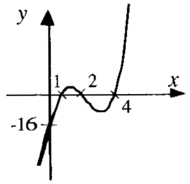
d.


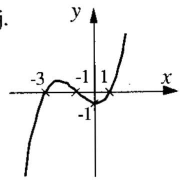
e.


f.


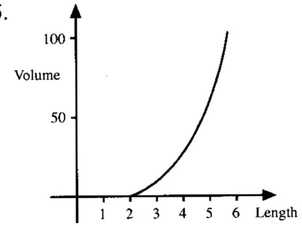
g.


h.


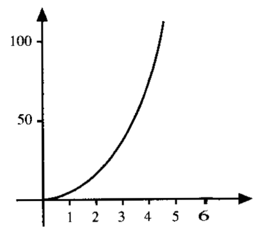
i.


j.


4. a. $y = (x+2)(x-1)(x-3)$ b. $y = (x+2)(x+1)(x-2)$
c. $y = -(x+1)(x-1)(x-3)$ d. $y = (x+3)(x-2)^2$
e. $y = -4(x+3)(x+2)(x-2)$ f. $y = -\frac{2}{3}(x+3)^2(x-2)$

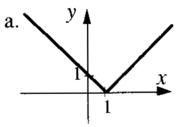
5.


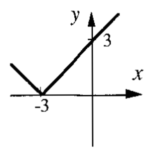
$V = (L-2)(L-1)L$

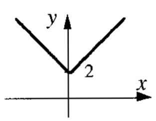
6.


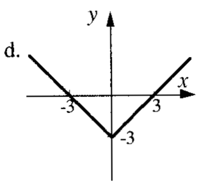
$V = L^2(L+1)$

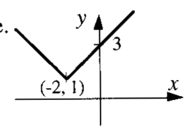
Exercise 13c

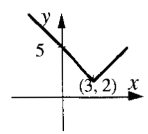
1. a.


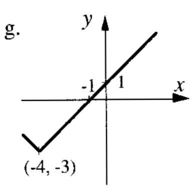
b.


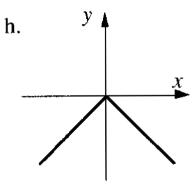
c.


d.


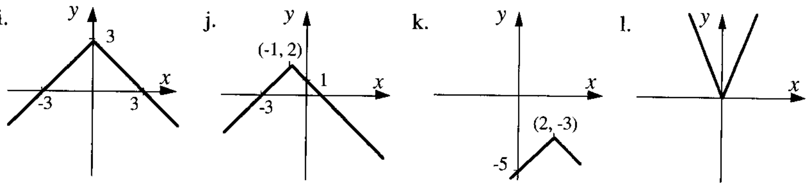
e.


f.


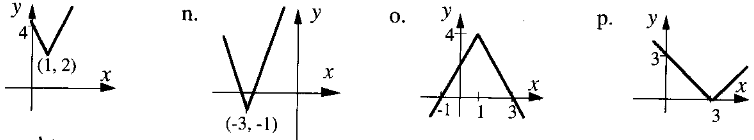
g.


h.


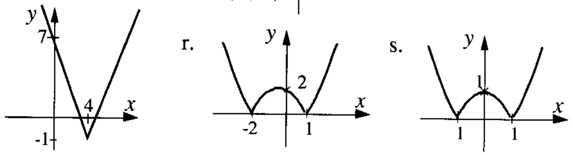
i. j. k. l.



m. n. o. p.



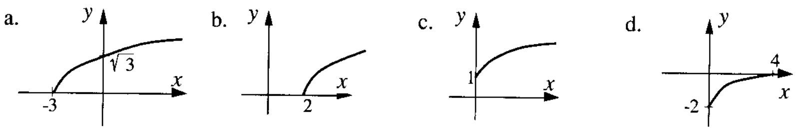
q. r. s.



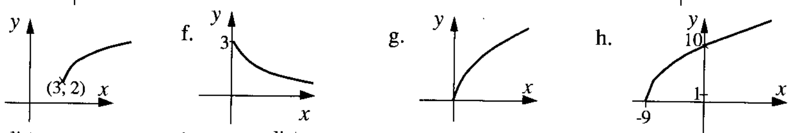
2. a. $y = |x + 1|$ b. $y = 2 - |x + 2|$ c. $y = 2|x - 2| + 1$ d. $y = 2|x|$
   e. $y = -1 - \frac{1}{4}|x - 4|$

Exercise 13d

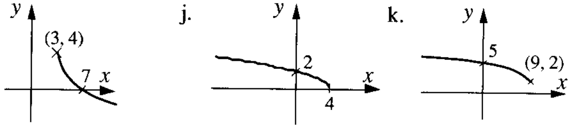
1. a. b. c. d.



e. f. g. h.



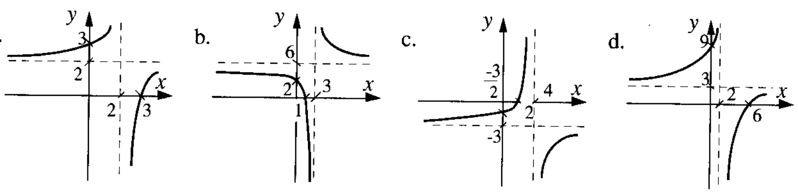
i. j. k.



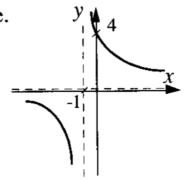
2. a. Domain $\{x: x \geq -3\}$ Range $\{y: y \geq 0\}$ b. Domain $\{x: x \geq 2\}$ Range $\{y: y \geq 0\}$
   c. Domain $\{x: x \geq 0\}$ Range $\{y: y \geq 1\}$ d. Domain $\{x: x \geq 0\}$ Range $\{y: y \geq -2\}$
   e. Domain $\{x: x \geq 3\}$ Range $\{y: y \geq 2\}$ f. Domain $\{x: x \geq 0\}$ Range $\{y: y \leq 3\}$
   g. Domain $\{x: x \geq 0\}$ Range $\{y: y \geq 0\}$ h. Domain $\{x: x \geq -9\}$ Range $\{y: y \geq 1\}$
   i. Domain $\{x: x \geq 3\}$ Range $\{y: y \leq 4\}$ j. Domain $\{x: x \leq 4\}$ Range $\{y: y \geq 0\}$
   k. Domain $\{x: x \leq 9\}$ Range $\{y: y \geq 2\}$

Exercise 13e

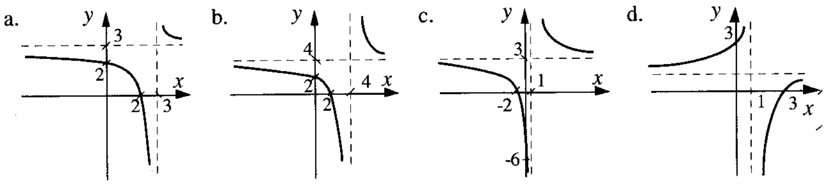
1. a. b. c. d.



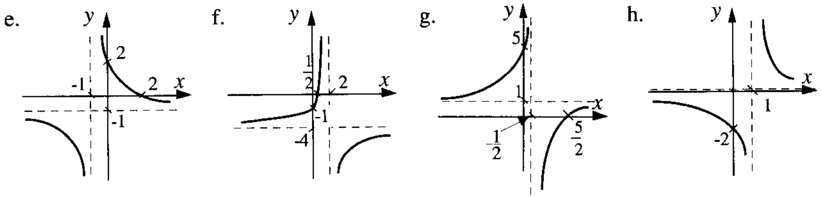
e.



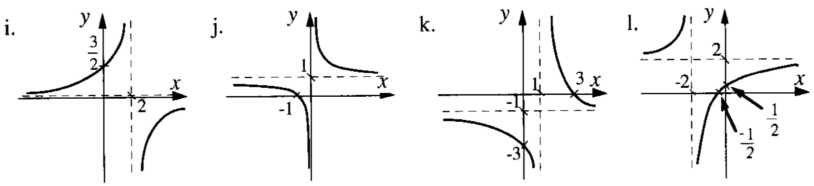
2. a. b. c. d.



e. f. g. h.



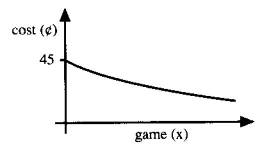
i. j. k. l.



3. a. Domain $\{x: x \neq 3\}$ Range $\{y: y \neq 3\}$ b. Domain $\{x: x \neq 4\}$ Range $\{y: y \neq 4\}$
   c. Domain $\{x: x \neq 1\}$ Range $\{y: y \neq 3\}$ d. Domain $\{x: x \neq 1\}$ Range $\{y: y \neq 1\}$
   e. Domain $\{x: x \neq -1\}$ Range $\{y: y \neq -1\}$ f. Domain $\{x: x \neq 2\}$ Range $\{y: y \neq -4\}$
   g. Domain $\{x: x \neq \frac{1}{2}\}$ Range $\{y: y \neq 1\}$ h. Domain $\{x: x \neq 1\}$ Range $\{y: y \neq 0\}$
   i. Domain $\{x: x \neq 2\}$ Range $\{y: y \neq 0\}$ j. Domain $\{x: x \neq 0\}$ Range $\{y: y \neq 1\}$
   k. Domain $\{x: x \neq 1\}$ Range $\{y: y \neq -1\}$ l. Domain $\{x: x \neq -2\}$ Range $\{y: y \neq 2\}$

4. a. $y = \frac{12x + 12}{x + 3}$ b. $y = \frac{3x - 6}{x}$ c. $y = \frac{3 - x}{x}$ d. $y = \frac{2x + 6}{x + 2}$ e. $y = \frac{-4 - 2x}{x - 5}$

5. a. $x + 10$
   b. $(x + 10)(y + 5)$
   c. $y = \frac{500}{x + 10} - 5$
   d.



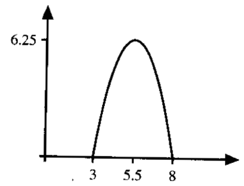
Chapter 13 – Problems and Investigations

1. $D = 5 + \frac{1}{2}(t-2)^2$

2.a.

BD(x)	AB	BC	Area of rectangle
3	5	0	0
$3\frac{1}{2}$	$4\frac{1}{2}$	$\frac{1}{2}$	2.25
4	4	1	4
$4\frac{1}{2}$	$3\frac{1}{2}$	$1\frac{1}{2}$	5.25
5	3	2	6
6	2	3	6
7	1	4	4
8	0	5	0

b.



c. $A = (8-x)(x-3)$ d. 6.25

3. 58.5 km 4. 486 5. a. $y = \frac{3}{2}|x-2|$ b. $y = 6 - \frac{3}{2}|x-6|$ c. At B

Chapter 14: Equation of a Circle

Exercise 14a

1. a. $x^2+y^2=81$ b. $x^2+y^2=64$ c. $x^2+y^2=36$ d. $x^2+y^2=25$ e. $x^2+y^2=20.25$
2. a. $x^2+y^2=36$ b. $x^2+y^2=1$ c. $x^2+y^2=\frac{1}{4}$ d. $x^2+y^2=0.09$
3. a. $\sqrt{13}$ b. $\sqrt{41}$ c. $\sqrt{29}$ d. 2 e. $\sqrt{5}$
4. a. $x^2+y^2=5$ b. $x^2+y^2=34$ c. $x^2+y^2=5$ d. $x^2+y^2=13$ e. $x^2+y^2=52$ f. $x^2+y^2=29$ g. $x^2+y^2=8.5$ h. $x^2+y^2=52$ i. $x^2+y^2=25$ j. $x^2+y^2=4$
5. a. $x^2+y^2=2$ b. $x^2+y^2=25$ e. $x^2+y^2=9$

Exercise 14b

1. $(x-2)^2+(y-3)^2=25$ 2. $(x+3)^2+(y-1)^2=16$ 3. $(x+2)^2+(y+3)^2=4$
4. $(x-1)^2+(y-4)^2=9$ 5. $(x-1)^2+(y-2)^2=17$ 6. $(x-4)^2+(y-2)^2=13$
7. $(x-\frac{1}{2})^2+(y-1\frac{1}{2})^2=\frac{50}{4}$ 8. $(x-1)^2+y^2=8$ 9. (-1, 3), 4
10. (2, 4), 5 11. (4, -1), 6 12. $(x-1)^2+(y-2)^2=25$
13. $(x-3)^2+(y-7)^2=100$ 14. $(x+5)^2+(y-2)^2=169$
15. $(x-2)^2+(y-3)^2=2$ 16. Shane

Exercise 14c

1. a. (0, 2) b. (1.73, 1), (-1.73, 1) c. (2, 0) d. (-1, 1.73), (-1, -1.73) e. (1.32, 1.5), (-1.32, 1.5) f. No intersection g. (1.41, 1.41), (-1.41, -1.41) h. (0.89, 1.79), (-0.89, -1.79) i. (0.63, 1.90), (-0.63, -1.90) j. (1.79, 0.89), (-1.79, -0.89)
2. a. (0.82, 1.82), (-1.82, -0.82) b. (1.82, 0.82), (-0.82, -1.82) c. (-1.27, -1.54), (0.47, 1.94) d. (0, 2), (-2, 0) e. (0, -2), (1.6, 1.2)

3. a. (2, 4), (-2, 4) b. (2, 4) c. No intersection d. (0, 0), (1, 1) e. (0, 0), (2, 4) f. (0, 0), (3, 9) g. (3, 9), (-1, 1) h. (1, 1) i. (-4, 16), (1, 1) j. (1.62, 2.62), (-0.62, 0.38)
4. a. $(\frac{1}{3}, 3)$ b. $(\frac{1}{5}, 5)$ c. $(2, \frac{1}{2})$ d. $(4, \frac{1}{4})$ e. (1, 1), (-1, -1) f. (0.71, 1.41), (-0.71, -1.41) g. (1, 1) h. $(\frac{1}{2}, 2)$, (-1, -1) i. (1.62, 0.62), (-0.62, -1.62) j. No intersection

Chapter 14 – Problems and Investigations

1. Each is a tangent 2. $\pm\sqrt{18}$ 3. $y = \frac{\sqrt{5}}{2}x + 3;\ y = -\frac{\sqrt{5}}{2}x + 3$ 4. $(\sqrt{2}, 2), (-\sqrt{2}, 2)$

Chapter 15: Indices

Exercise 15a

1. a. 0.0625 b. 0.4019 c. 7.3046 d. 5.0027 e. 4.1890 f. 9 g. 27.648 h. 0.0865 i. 1.06 j. -1.2022 k. 0.2712
2. a. $\frac{1}{9}$ b. $\frac{1}{64}$ c. $\frac{1}{16}$ d. $\frac{27}{64}$ e. $\frac{8}{25}$ f. 9 g. 16 h. $\frac{25}{9}$ i. $\frac{125}{64}$ j. $\frac{27}{4}$ k. $\frac{25}{27}$ l. $\frac{1}{24}$ m. $\frac{1}{10}$ n. $\frac{7}{5}$ o. $\frac{8}{3}$ p. $\frac{1}{6}$ q. $\frac{1}{15}$ r. $\frac{1}{12}$ s. 10 t. 12 u. $\frac{1}{36}$ v. $\frac{1}{175}$ w. 2 x. 4 y. $\frac{1}{14}$ z. $\frac{4}{3}$
3. a. $\frac{3}{2}$ b. $\frac{1}{36}$ c. 144 d. $\frac{1}{576}$ e. $\frac{5}{6}$ f. $\frac{5}{18}$ g. $\frac{33}{10}$ h. $\frac{1}{8}$ i. $\frac{1}{2}$ j. $\frac{17}{12}$ k. $-\frac{73}{18}$ l. $\frac{6}{5}$ m. $\frac{5}{36}$ n. $\frac{11}{16}$ o. $\frac{11}{4}$
4. a. $\frac{1}{2}$ b. $\frac{1}{6}$ c. 2 d. $\frac{5}{9}$ e. $\frac{3}{2}$ f. $\frac{2}{3}$ g. 9 h. $\frac{3}{2}$ i. $1\frac{3}{5}$
5. a. 3×2^2 b. $2^3\times5^2$ c. 5×3^2 d. $5^2\times2^{-4}$ e. 2×3^{-1} f. $2^2\times3^{-3}$ g. $2^4\times3^{-4}$ h. $2^6\times3^{-4}$ i. $3^2\times2^{-2}$

Exercise 15b

1. a. $\frac{1}{x^3}$ b. $\frac{1}{y^2}$ c. $\frac{1}{a^5}$ d. $\frac{1}{b^4}$ e. $\frac{1}{z^7}$ f. $\frac{1}{y^6}$ g. $\frac{1}{z^{11}}$ h. $\frac{1}{xy}$ i. $\frac{1}{abc}$ j. $\frac{1}{x^2y^2}$ k. $\frac{1}{2x}$ l. $\frac{1}{4x^2}$ m. $\frac{1}{9a^2}$ n. $\frac{1}{8x^3}$ o. $\frac{3}{x}$ p. $\frac{a}{b^3}$ q. $\frac{b^2}{a^3}$ r. $\frac{1}{a}$ s. $\frac{1}{2x}$ t. $\frac{1}{4x^3}$ u. $\frac{3}{5x^2}$ v. $\frac{2}{7a^3}$ w. $\frac{4}{9a^2}$ x. $\frac{3}{5x^4}$ y. $\frac{4}{5x^3}$

2. a. x^{-6} b. y^{-5} c. z^{-2} d. $4x^{-1}$ e. $5y^{-2}$ f. $7z^{-5}$
g. $\frac{x^{-2}}{2}$ h. $\frac{a^{-4}}{3}$ i. $\frac{y^{-4}}{5}$ j. $\frac{2x^{-2}}{5}$ k. $\frac{3y^{-3}}{4}$ l. $\frac{3x^{-2}}{4}$
m. $\frac{4y^{-2}}{25}$ n. $\frac{x^{-1}}{8}$ o. $\frac{x^{-3}}{8y^{-2}}$

3. a. $\frac{y^3}{2x^2}$ b. $\frac{2b^4}{3a^3}$ c. $\frac{x}{5}$ d. $\frac{2y^4}{x^2}$ e. $\frac{x^5}{y}$ f. $\frac{2y^3}{x}$
g. $\frac{8}{x^7y^2}$ h. $\frac{1}{25x^3}$ i. $\frac{1}{6xy}$ j. $\frac{1}{27x^4}$ k. $\frac{x}{2y^2}$ l. $\frac{x^2}{9y^3}$
m. $\frac{2}{3x^2}$ n. $\frac{4y^2}{5x^3}$ o. $\frac{x+y}{xy}$ p. $\frac{x+1}{x^2}$ q. $\frac{xy^2+2}{2y}$

Exercise 15c

1. a. $a^{\frac{1}{2}}$ b. $c^{\frac{1}{2}}$ c. $x^{\frac{1}{3}}$ d. $a^{\frac{1}{5}}$ e. $u^{\frac{1}{6}}$ f. $x^{\frac{2}{5}}$
g. $x^{\frac{3}{7}}$ h. $a^{\frac{3}{8}}$ i. $x^{-\frac{1}{2}}$ j. $y^{-\frac{1}{2}}$ k. $a^{-\frac{1}{3}}$ l. $x^{-\frac{1}{5}}$
m. $x^{-\frac{2}{7}}$ n. $x^{-\frac{5}{9}}$ o. $a^{-\frac{3}{7}}$ p. $a^{-\frac{8}{9}}$ q. $4a^{\frac{1}{3}}$ r. $6y^{-\frac{1}{3}}$
s. $8x^{-\frac{7}{3}}$ t. $\frac{x^{-\frac{2}{3}}}{7}$ u. $\frac{2x^{-\frac{3}{4}}}{5}$ v. x^{-1} w. $x^{\frac{1}{6}}$ x. $a^{-\frac{5}{3}}$ y. $x^{-\frac{1}{12}}$

2. a. $\sqrt[7]{a^3}$ b. $\sqrt[4]{b^3}$ c. $\sqrt[5]{x^2}$ d. $\frac{1}{\sqrt[3]{x^2}}$ e. $\frac{1}{\sqrt[4]{y^3}}$

3. a. $(xy)^{\frac{1}{2}}$ or $(x)^{\frac{1}{2}}(y)^{\frac{1}{2}}$ b. $(x)^{\frac{1}{2}}y$ c. $x(y)^{-\frac{1}{2}}$ d. $x^2(y)^{-\frac{1}{3}}$ e. $2x(y)^{-\frac{3}{5}}$
f. $(x)^{-\frac{5}{4}}y$ g. $(x)^{\frac{1}{2}}(a)^{\frac{1}{3}}$ h. $(a)^{-\frac{1}{2}}(c)^{\frac{1}{2}}$ i. b

Exercise 15d

1.	16	2.	0.11	3.	82.19	4.	-32	5.	2.25	6.	1.32
7.	-2.11	8.	No sol.	9.	3.51	10.	1720.51	11.	2.73	12.	1.83
13.	74.30	14.	-0.5109	15.	1.25	16.	2.1715	17.	14.64	18.	0.0833
19.	No sol.	20.	0.0638								

Exercise 15e

1. a. 4 b. 8 c. 81 d. 25 e. 32 f. 0.2 or $\frac{1}{5}$
g. $\frac{1}{4}$ or 0.25 h. 81 i. 32 j. 8 k. $\frac{1}{8}$ l. $\frac{1}{8}$
m. 4 n. 8 o. 5 p. 27 q. $\frac{1}{27}$ r. $\frac{1}{4}$
s. $\frac{1}{9}$ t. $\frac{1}{243}$

2. a. 64 b. 2 381 c. 0.20 d. 3.04 e. No Sol. f. -10.08
g. 6.55 h. 270.8 i. 1.67

3. a. \$51 b. 81.51 m 4. a. 14336 b. 4 c. 1.8

Chapter 15 – Problems and Investigations

1. This is false 2. This is true

Chapter 16: Logarithms

Exercise 16a

1. a. 3.24 b. 4.92 c. 5.83 d. 2.42 e. 9.45 f. 14.49
g. 19.47 h. 24.61 i. 0.56 j. 0.91 k. 7.04 l. No Sol.
m. 3.78 n. 1.22 o. 0.79 p. 3.51 q. 2.87 r. 9.61
s. 2.49 t. No Sol. u. 0.88 v. 1.30 w. 0.45 x. 1.36
y. No Sol.

2. a. 0.88 b. 1.74 c. 1.34 d. -2.27 e. 0.16 f. 0
g. 2.12 h. 0.59 i. 1.45 j. 0.51

Exercise 16b

1. a. *ln* 24 b. *ln* 30 c. *ln* 24 d. *ln* 13 e. *ln* 2 f. *ln* 25
g. *ln* 32 h. *ln* 6 i. *ln* 21 j. *ln* 5

2. a. $x + y - 2z$ b. $3x + 2y - z$ c. $\frac{1}{3}x$ d. $x + \frac{2}{3}y - z$

Exercise 16c

1. a. 3.669 b. 16.445 c. 0.045 d. 0.323 e. 8.110 f. 75.077
g. 148.916 h. 21.964 i. 5.623 j. 0.050 k. 0.271 l. 7.021
m. 1.396 n. 2.117 o. 0.607

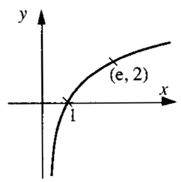
2. a. 3.67 b. 0.09 c. 1.40 d. 9.36 e. 7.13 f. 26.64
g. 5.26 h. 1.24 i. 35.84 j. 0.46

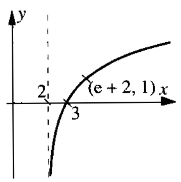
3. a. 40.4473 b. 56.2609 c. 15.3329 d. 0.0018 e. 0.7118

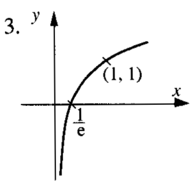
4. a. 5 b. 0.41 c. 1.73 d. 0.13 e. 2

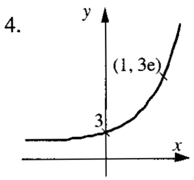
5. a. e^{a+b} b. e^{x+2} c. e^{x-3} d. e^x e. e^{3x-3} f. e^{y-x}
g. e^{2x} h. e^{-3x} i. e^{x-2} j. e^{x+2y}

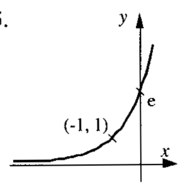
Exercise 16d

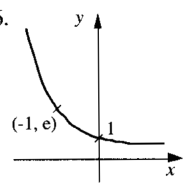
1.


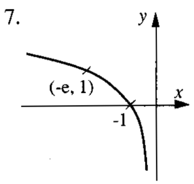
2.


3.


4.


5.


6.


7.


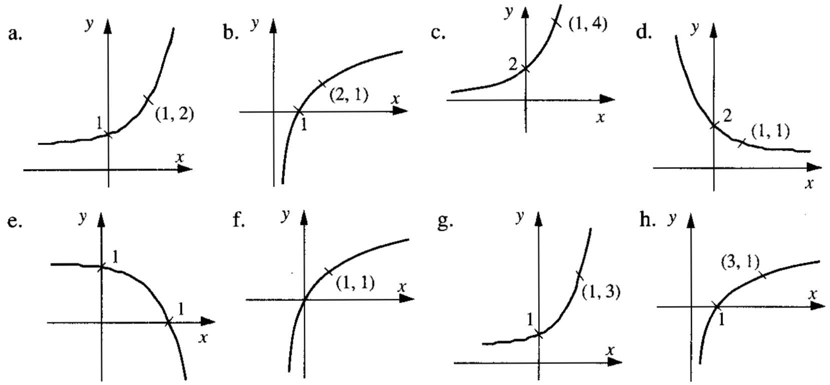
Exercise 16e

1. a. 1.39 b. -1.20 c. 1.18 d. -0.80 e. -0.53 f. -1.22
2. a. 20.09 b. 0.14 c. -0.86 d. 27.67 e. 0.67

Exercise 16f

1. 2.10 2. 199.53 3. 2508.52 4. 0.04 5. 3.68

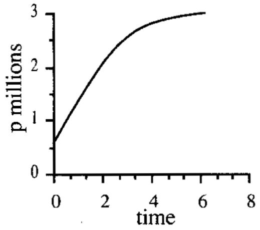
Exercise 16g

1.


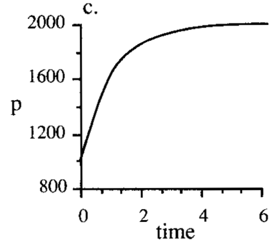
2. a. $\log_2 8 = 3$ b. $\log_3 81 = 4$ c. $\log_5 25 = 2$ d. $\log_6 216 = 3$ e. $\log_x b = a$
3. a. $3^2 = 9$ b. $(9)^{\frac{1}{2}} = 3$ c. $25^{1.5} = 125$ d. $(1000)^{\frac{1}{3}} = 10$ e. $3^{-1} = \frac{1}{3}$
4. a. 6 b. -1 c. 6 d. 7 e. -1 f. -3
   g. $\frac{1}{2}$ h. 3 i. $\frac{1}{2}$ j. $\frac{1}{4}$ k. $\frac{1}{2}$ l. $\frac{3}{2}$
   m. 2 n. $\frac{5}{6}$ o. $-\frac{1}{2}$ p. 1 q. -1 r. -3
   s. $\frac{5}{3}$ t. $\frac{3}{4}$
5. a. 0.954 b. -2.117 c. 2.510 d. 2.885 e. 0.972 f. 1.262
   g. 1.390 h. 1.710 i. 2.767 j. 2.484
6. a. 1 b. 1 c. 3 d. $\frac{1}{2}$ e. $\frac{1}{4}$ f. $\frac{5}{4}$
   g. $\frac{3}{4}$ h. $\frac{1}{2}$ i. $\frac{4}{5}$ j. -1
7. a. 64 b. 81 c. 2 d. 216 e. 1.73 f. 64
   g. 81 h. 3 i. 2 j. 36 k. 5 l. 1
   m. -3 n. $\frac{1}{3}$ o. $\frac{2}{3}$
8. a. $\log xyz$ b. $\log x^2y$ c. $\log xy^2$ d. $\log \frac{xy}{z}$ e. $\log \frac{ab^3}{c^2}$ f. $\log \frac{(p)^{\frac{1}{2}}}{(q)^{\frac{3}{2}}}$
   g. $\log xyz$ h. $\log y$ i. $\log xy^4$ j. $\log y$ k. $\log \frac{1}{y^2}$ l. $\log y^2$
   m. $\log xy$ n. $\log \frac{1}{z^2}$ o. $\log xy^3$ p. $\log_a x^2a^5$

9. a. log60 b. log12 c. log75 d. log12 e. log24 f. $\log \frac{6}{7}$
   g. log 1 h. $\log \frac{25}{3}$ i. log96 j. log12
10. a. 3 b. $\frac{2}{3}$ c. $\frac{2}{3}$ d. $\frac{4}{3}$ e. $\frac{6}{5}$ f. 2
    g. $\frac{11}{5}$ h. 1 i. 1 j. 2

Chapter 16 – Problems and Investigations

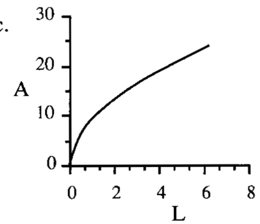
1. a. 571 429
   b. 2 623 653
   c. 1.79 years
   d.


   e. 3 million

2. a. 1 000
   b. 1 969
   c.


   d. 2000
   e. 0.69 years

3. a. A = 100 000
   K = 0.03922
   b. $148 023
   c. 58.7 years later

4. a. K = 4.5 b. $A = K^{\frac{3}{2}}L^{\frac{1}{2}}$ c.


   d. 44.95 e. $W = \frac{K^{\frac{3}{2}}}{L^{\frac{1}{2}}}$

5. The height of the chimney is 10m.

Chapter 17: Sequences and Series

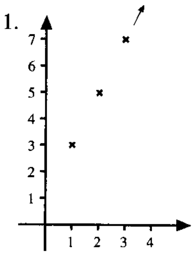
Exercise 17a

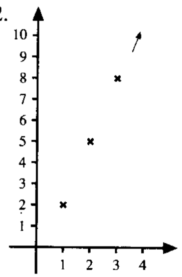
1. a. 6, 7, n + 1 b. 4, 5, n – 1 c. 15, 18, 3n d. 13, 15, 2n + 1
   e. 5, 7, 2n – 5 f. 16, 32, 2^{n-1} g. 81, 243, 3^{n-1} h. 162, 486, $2(3^{n-1})$
   i. $\frac{4}{5}, \frac{5}{6}, \frac{n-1}{n}$ j. $\frac{5}{6}, \frac{6}{7}, \frac{n}{n+1}$ k. $\frac{4}{5}, \frac{5}{6}, \frac{n-1}{n}$ l. 4, 2, 12 – 2n
   m. 1, $\frac{1}{2}$, $16(2^{1-n})$ n. 0.20, 0.17, $\frac{1}{n}$ (to 2 d. p.) o. 25, 36, n^2
   p. 15, 18, 3n q. $\frac{1}{25}, \frac{1}{36}, \frac{1}{n^2}$ r. 11, 13, 2n – 1 s. 26, 37, $n^2 + 1$
   t. 2.24, 2.45, $\sqrt{n}$ (to 2 d.p)
2. a. 1, 2, 3, 4, n b. 3, 4, 5, 6, n + 2 c. 3, 5, 7, 9, 2n + 1
3. a. 6, 8, 10, 12, 2n + 4 b. 5, 6, 7, 8, n + 4 c. 9, 12, 15, 18, 3n + 6

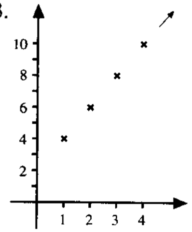
Exercise 17b

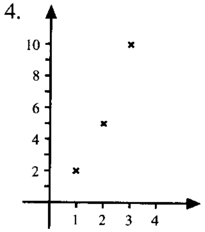
1. a. 2, 5, 10, 17 b. 5, 7, 9, 11 c. 1, 8, 27, 64 d. 3, 7, 13, 21
   e. $2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}$ f. 2, 2.25, 2.37, 2.44 g. $-\frac{1}{2}, \frac{1}{2}, -\frac{3}{8}, \frac{1}{4}$ h. 0, 5, 8, 17
2. a. 30, 42, 58.8, 82.32 b. 12, 13.2, 14.52, 15.972
   c. 41, 48, 55, 62 d. $\sqrt{2}$, 2, 4, 16 e. 6, 12, 24, 48
3. a. 1, 2, 4, 8 b. 1, 5, 9, 13 c. 3, 9, 81, 6561 d. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}$
   e. 1, 3, 7, 15

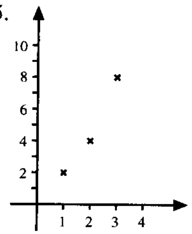
Exercise 17c

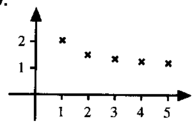
1.


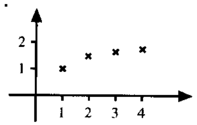
2.


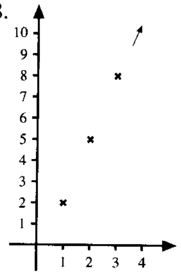
3.


4.


5.


6.


7.


8.


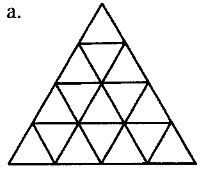
Exercise 17d

1. a. No limit b. 0 c. 1 d. 0 e. 0
   f. No limit g. 1 h. No limit i. $\frac{1}{2}$ j. $\frac{1}{2}$
2. a. 91, 85, 81, $78\frac{1}{3}$, $76\frac{5}{9}$, $75\frac{10}{27}$ b. 73
   c. As a consequence of her diet she loses 27 kg.
3. a. n + 2 b. 8n + 50 c. $\frac{8n + 50}{n + 2}$
   d. As n gets very large the 50 and 2 in the formula become negligibly small.

Exercise 17e

1. a. 13 b. 5.4321 c. $\frac{47}{60}$ d. 0 e. 34 f. 45
   g. 49 h. 2 500
2. a. $\sum_{n=1}^{3} 3n$ b. $\sum_{n=2}^{4} 2n$ or $\sum_{n=1}^{3} 2n + 2$ c. $\sum_{n=5}^{8} n$ d. $\sum_{n=1}^{4} n^2$ e. $\sum_{n=1}^{4} 2^{n+1}$

Exercise 17f

1. a. 17, 59, 3n – 1 b. 35, 51, 4n + 7 c. -6, -34, 6 – 2n d. $3\frac{1}{2}, 6\frac{1}{2}, \frac{n+1}{2}$
   e. 2.5, 3.1, 0.1n + 1.9
2. a. -4, 0, 4, 8, 12, 16, 4n-8 b -14, -9, -4, 1, 6, 11, 16, 21, 5n-19
   c. -1, 1, 3, 5, 7, 9, 11, 13, 2n-3 d. 12, 9, 6, 3, 0, -3, -6, -9, 15-3n
   e. -201 f. 41st term g. 14th term h. -51 i. 276 j. 13
3. a. 135 months b. i. After 20 months ii. Sept. 1993 c. After 100 months
4. a.


   b. L_n: 1, 3, 5, 7, 9, 2n – 1
   J_n: 1, 4, 9, 16, n^2
   S_n: 1, 2, 3, 4, n
   P_n: 2, 3, 4, 5, n + 1
   W_n: 3, 6, 10, 15, $\frac{(n+1)(n+2)}{2}$
   c. L_n, S_n, P_n

Exercise 17g

1. 610 2. 60 300 3. 6 475 4. -967.5 5. 477.5 6. 999
7. 1 524 8. 10 822.5 9. 3 925 10. -12 570 11. 5 406 12. 1 113
13. 11 14. 5 15. 2 16. 1 944 17. 4 242 18. 1 092
19. 85.44 kg 20. Some time between 35 and 40 minutes after the doors opened.

Exercise 17h

1. a. $\frac{1}{8}, \frac{1}{128}, 4\left(\frac{1}{2}\right)^{n-1}$ b. 1 944, 52 488, $8(3)^{n-1}$ c. 512, -4 096, $-(-2)^{n-1}$
   d. $\frac{-8}{243}, \frac{-128}{1\,594\,323}, 3\left(-\frac{2}{9}\right)^{n-1}$ e. $93\frac{3}{4}, 2\,343\frac{3}{4}, \frac{3}{4}(5)^{n-1}$
2. a. $\frac{5}{9}, \frac{5}{3}$, 5, 15, 45, 135, $\frac{5}{9}(3)^{n-1}$ b. 3, 6, 12, 24, 48, 96, $3(2)^{n-1}$ c. $1\frac{7}{9}, 20\frac{1}{4}, \frac{32}{27}(1.5)^{n-1}$
   d. $\frac{54}{125}, \frac{54}{125}\left(\frac{5}{3}\right)^{n-1}$ e. $\frac{4}{729}, 4\left(\frac{1}{3}\right)^{n-1}$ f. $\frac{11}{8}, \frac{11}{8}(2)^{n-1}$
   g. $-12, 24\left(-\frac{1}{2}\right)^{n-1}$ h. 6 i. Eighth term
   j. Ninth term k. Tenth term l. 3.6
3. a. 5 000, 5 550, 6 160.50, 6 838.16, 7590.35 b. 52 years 4. 3 170 000 (3 sf)
5. a. $732 b. The company would sell the computer during the sixth year (Between the end of the fifth and the end of the sixth).

Exercise 17i

1. 3 069 2. 113.33 3. 4 4. 5 5. 453.32 6. 44
7. $608 171 8. a. 147.75 kg, 145.53 kg, 143.35 kg, 141.20 kg, 139.08 kg b. 10.92 kg
9. 522 tonnes 10. 23 875 11. 3 840.90 12. 4 404.13 13. 11 14. 10

Exercise 17j

1. b, c, e
2. a. 3 b. 6 c. $4\frac{2}{7}$ d. $-2\frac{10}{11}$ e. $\frac{2}{3}$
3. a. $12\frac{1}{2}$ b. $10\frac{10}{13}$ c. $40\frac{1}{2}$ d. 172.23 e. 32

4. $\frac{3}{8}$ 5. 75 000 6. \$4 000
7. a. \$406 500, \$203 250, \$101 625 b. \$813 000
8. a. 19 683, 6 561, 2 187, 729, 243 b. 29 524.5 9. 1 300 000
10. a. 1 g b. 5, 1, 0.2, 0.04, 0.008
 c. Each term when divided by its predecessor gives a constant ratio of 0.2
 d. $5(0.2)^{n-1}$ g e. 6.25 g

Chapter 17 – Problems and Investigations

1. Take the six month plan 2. During the ninth year

Chapter 18: Gradient Functions

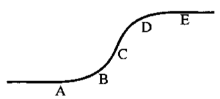
Exercise 18a

1. a. 2	b. -6	2. a. 0	b. -4	3. a. 4	b. -2
4. a. 3	b. 3	c. 3	5. a. -2	b. -2	c. -2
6. a. 0	b. 0	c. 0	7. a. 12	b. 27	c. 3
8. a. 5	b. 9	c. -11			

Exercise 18b

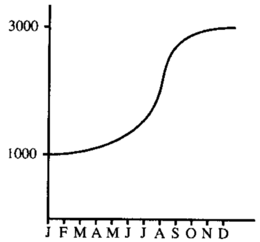
1. a. Moving at constant speed till it passes A, after which it notices an increasing 'slowing effect' till C is passed, after which the slowing effect diminishes till E is reached, at which time the slowing effect disappears.
 b. C
2. a. Moving at constant speed till A is passed after which the car slowly slows down till just after C it rapidly slows. Once past D this slowing effect quickly disappears.
 b. D

3.



4.

5.



Exercise 18c

1. a.

x	-3	-2	-1	0	1	2	3
y	-27	-8	-1	0	1	8	27
gradient	27	12	3	0	3	12	27

 b. gradient = $3x^2$

2. a.

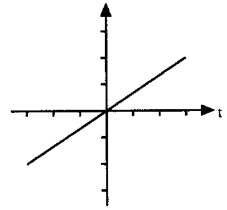
x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81
gradient	-108	-32	-4	0	4	32	108

 b. gradient = $4x^3$

3. a. gradient = $5x^4$ b. gradient = $4x$ c. gradient = $2x - 1$
 d. gradient = $2x + 4$ e. gradient = 2 f. gradient = 0

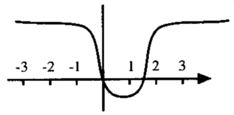
Exercise 18d

1. a.



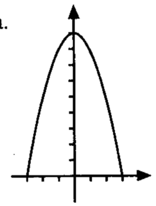
 b. The rate of change of distance from the house changes at a steady rate.

2. a.

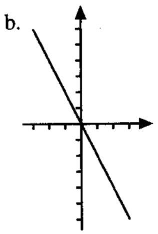


 b. The rate at which profits increase is fairly constant until June. It decreases sharply until near the middle of August at which time it quickly recovers to its pre-June value.

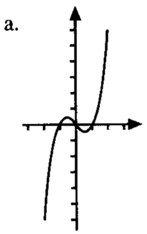
3. a.



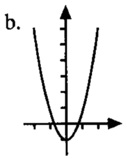
 b.



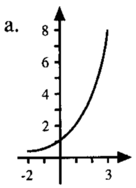
4. a.



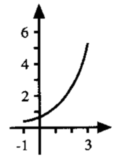
 b.



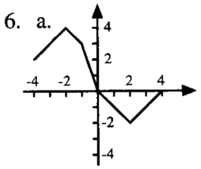
5. a.



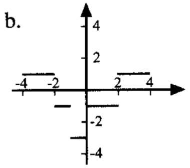
 b.



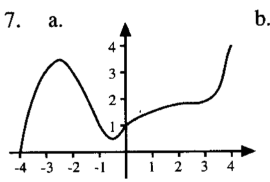
6. a.



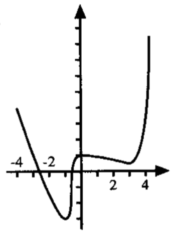
 b.



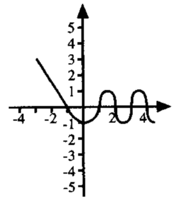
7. a.



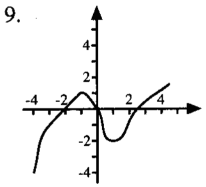
 b.



8.

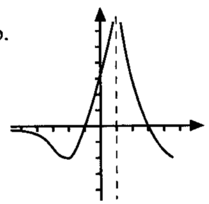


9.



Chapter 18 – Problems and Investigations

1. b.



2. a. $-\frac{1}{x^2}$ b. $\frac{1}{2\sqrt{x}}$ c. e^x d. $\frac{1}{x}$ e. $\frac{1}{x \ln 10}$ f. $-\frac{1}{(x+2)^2}$
g. $\begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$ h. $1 - \frac{1}{x^2}$ i. $1 + \ln x$ j. $(1 + x)e^x$

Chapter 19: Derivatives

Exercise 19a

1. a. $5x^4$ b. $22x^{21}$ c. $48x^{47}$ d. $21x^6$ e. $45x^8$ f. $20x$
g. $\frac{2x}{3}$ h. $24x^2 - 6x$ i. $3x^5$ j. $\frac{8x}{3}$ k. 0
l. 5 m. $4x + 3$ n. $70x^9 - \frac{96x^{11}}{5}$ o. $2x + 2$ p. $8x + 4$
q. 3 r. $4x^3 + 6x^2 + 2x$ s. $8x - 12$ t. $2x + 3$ u. $2x + 5$
v. $3x^2 + 6x + 1$ w. $5x^4 + 12x^3 + 6x^2$ x. $4x^3 + 6x^2 + 6x + 2$
y. $4x + 6$ z. $4x^3 - 5x + \frac{3}{2}$
2. a. $12x^2$ b. $12x$ c. $10x + 3$ d. $9t^2 + 8t - 3$ e. $4u + 9$
f. $6x$ g. 2 h. $20x^4 + \frac{1}{7}$ i. $6r^2 + 16r + 6$ j. $18t^2 + 16t + 9$
3. a. i. 7 ii. 10 iii. $\frac{1}{2}$ b. i. 3 ii. 2 iii. 4 iv. $3\frac{1}{2}$
c. i. 125 ii. 17 iii. 0, 1 d. i. -6 ii. 9 iii. 0 iv. $-\frac{1}{6}$
e. i. $-15\frac{1}{3}$ ii. -9 iii. -2, 4 iv. -3, 5

Exercise 19b

1. $-4x^{-5}$ 2. $-5x^{-6}$ 3. $-7u^{-8}$ 4. $-12u^{-5}$ 5. $-30x^{-7}$
6. $-\frac{10}{3}y^{-6}$ 7. $-\frac{3}{2}x^{-3}$ 8. $-\frac{4}{5}x^{-5}$ 9. $-\frac{6}{7}x^{-3}$ 10. $-6x^{-4} - 12x^{-5}$

Exercise 19c

1. a. $\frac{1}{4}(x)^{-\frac{3}{4}}$ b. $\frac{1}{5}(r)^{-\frac{4}{5}}$ c. $\frac{2}{3}(u)^{-\frac{1}{3}}$ d. $\frac{4}{5}(t)^{-\frac{1}{5}}$ e. $\frac{5}{3}(y)^{\frac{2}{3}}$
f. $-\frac{1}{2}(x)^{-\frac{3}{2}}$ g. $-\frac{3}{2}(u)^{-\frac{5}{2}}$ h. $-3(r)^{-\frac{7}{4}}$ i. $-\frac{25}{6}(u)^{-\frac{11}{6}}$ j. $-\frac{1}{6}(x)^{-\frac{3}{2}}$
k. $-\frac{2}{15}(x)^{-\frac{5}{3}}$ l. $-\frac{1}{2}(x)^{-\frac{7}{4}}$ m. $-\frac{12}{35}(x)^{-\frac{11}{7}}$ n. $-(x)^{-\frac{5}{2}}$ o. $-\frac{35}{3}(x)^{-\frac{10}{3}}$
2. a. $\frac{1}{4}$ b. $-\frac{3}{16}$ c. $-\frac{8}{243}$ d. $-\frac{1}{108}$ e. $\frac{2}{9}$
f. $-\frac{2}{9}$ g. $\frac{1}{15}$ h. $U'(27)$ i. $S'(16)$
j. $g'(4) = \frac{1}{4}$, $h'(4) = -\frac{1}{16}$

Chapter 19 – Problems and Investigations

1. a. $A = 3, B = \frac{1}{2}$ b. $A = 2, B = \frac{1}{3}$ 2. $a = 5, b = 2$

Chapter 20: Calculus and Tangents

Exercise 20

1. a. 5 b. 0 c. -3 d. 1 e. $\frac{1}{2}$
2. a. $y = 4x - 4$ b. $y = -4x - 2$ c. $y = 3x - 4\frac{1}{2}$ d. $y = \frac{3}{2}x$ e. $y = \frac{1}{4}x + 1$
3. a. $y = -\frac{1}{4}x + 4\frac{1}{2}$ b. $y = -4x + 18$ c. $y = -x$
4. a. $(1\frac{1}{2}, 2\frac{1}{4})$ c. $(2, 2\frac{2}{3})$, $(-2, -2\frac{2}{3})$
5. a. $(1, 1)$ b. $(-\frac{1}{4}, -2\frac{15}{16})$ c. $(4, 2)$

Chapter 20 – Problems and Investigations

1. $(2)^{-\frac{3}{2}}$ 2. $\frac{1}{3}$ 3. $C = \frac{1}{2}, D = 3$

Chapter 21: Increasing and Decreasing Functions: Turning Points

Exercise 21

1. $(2, -3)$ minimum point; increasing $x > 2$; decreasing $x < 2$
2. $(-4, 22)$ maximum point; increasing $x < -4$; decreasing $x > -4$
3. $(\frac{3}{4}, 6\frac{1}{8})$ maximum point; increasing $x < \frac{3}{4}$; decreasing $x > \frac{3}{4}$
4. $(-1, 5\frac{1}{3})$ maximum point; $(-3, 4)$ minimum point; increasing for $-3 < x < -1$; decreasing for $x < -3$, for $x > -1$.
5. $(-3, 12\frac{1}{2})$ minimum point; $(-2, 12\frac{2}{3})$ maximum point; increasing for $-3 < x < -2$; decreasing for $x < -3$ and for $x > -2$
6. $(-1, -9)$ minimum point; $(-3, -1)$ maximum point; increasing for $x < -3$, for $x > -1$; decreasing for $-3 < x < -1$
7. $(-2, 18)$ maximum point; $(2, -14)$ minimum point; increasing for $x < -2$, for $x > 2$; decreasing for $-2 < x < 2$
8. $(-3, -52)$ minimum point; $(3, 56)$ maximum point; increasing for $-3 < x < 3$; decreasing for $x < -3$, for $x > 3$
9. a. Cubic: x-intercepts, -1, 1; y-intercept 1; x-axis only touched at $x = 1$
b. $3x^2 - 2x - 1$ c. $x = -\frac{1}{3}$, maximum point; $x = 1$, minimum point.
d. $-\frac{1}{3} < x < 1$ e. 1 f. -1 g. $y = -x + 1$ h. 1
10. a. 3 b. $3x^2 - 6x - 9$ c. $x = -1, 3$ d. $-1 < x < 3$
e. $3 < c < 35$ f. $y = -12x + 31$

Chapter 21 – Problems and Investigations

1. $A = -21, B = 28$ 2. $A = 4, B = -6$ 3. $A = 2, B = -6, C = 3$

Chapter 22: Applications of Differentiation

Exercise 22a

1. a. 50 kg b. 1 037 kg (to nearest kg) c. 2.705 kg per day
d. 1.046 kg per day e. 2.25 kg per day

2. a.

t	0	10	20	30	40	50	60
s	2	82	262	602	1 162	2 002	3 182

b. i. 8 ms^{-1} ii. 18 ms^{-1} iii. 20 ms^{-1} iv. 53 ms^{-1}
c. i. 5 ms^{-1} ii. 12 ms^{-1} iii. 44 ms^{-1} iv. 137 ms^{-1}

3. b. 2.984 tonnes c. Decrease of 0.029 tonnes per day
d. 0.026 tonnes per day e. 0.082 – 0.002t tonnes per day
f. 0.06 tonnes per day g. 41st day

4. a. 6 cm b. 0.1 cm per minute c. $25 + t + 0.01t^2$ cm^2
d. 29.16 cm^2 e. 181.5 cm^2 f. $(5 + 0.1t)^3$
g. 166.375 cm^3 h. $7.5 + 0.3t + 0.003t^2$ cm^3 per minute

5. a. 3 m b. 0.1 m per hour c. 0.2 m per hour
d. After 20 hours e. $3 + 0.7t + 0.02t^2$ f. 5.28 m^2
g. 0.7 + 0.04t m^2 per hour h. After 107.5 hours

6. a. $\frac{db}{dt} = -0.1$ m per hour, $\frac{dh}{dt} = 0.2$ m per hour.
b. Each hour the base decreases by 0.1 m and the height increases by 0.2 m
c. $16\frac{2}{3}$ hours d. After 70 hours e. $14 + 1.2t - 0.02t^2$
f. 1.2 – 0.04t g. 1.2 m^2 per hour h. 15 hours
i. $37\frac{1}{2}$ hours. j. 32 m^2

7. a. 0.3 cm per minute b. 160 minutes c. $93\frac{1}{3}$ minutes.
d. $\pi(2 + 0.3t)^2$ e. 1 256 cm^2 [taking $\pi = 3.14$]
f. 71.59 cm^2 per minute

8. a. $M = 100\,000 - 10\,000t + 250t^2$ b. \$81 000
c. 500t – 10 000 d. For the first 20 months e. 0

9. a. $N = 60\,000 - 5\,000t + 100t^2$ b. \$45 900
c. -\$2 500 d. 18 months, 32 months
e. Decreasing by \$4 000 a month f. At 25 months

10. a. 0.1 m per min, 0.2 m per min b. This will never occur
c. $\frac{1}{2}(2 + 0.1t)(3 + 0.2t)$ d. 0.35 + 0.02t e. After 87.5 minutes
f. $\sqrt{(2 + 0.1t)^2 + (3 + 0.2t)^2}$ g. 1.6 + 0.1t

11. a. 5 – 2t b. After $2\frac{1}{2}$ minutes c. 27 cm^3 d. After $1\frac{1}{2}$ minutes
e. $6(5 - 2t)^2$ f. -24(5 – 2t) g. $-6(5 - 2t)^2$

Exercise 22b

1. 7 200 m 2. 5 400 litres after 30 minutes
3. Minimum is 15, maximum is 162
4. High tide when t = 5, with a depth of $216\frac{2}{3}$ cm above mark
Low tide when t = 19, with a depth of $240\frac{2}{3}$ cm below mark
5. 1250 m^2 6. a. $12\frac{1}{2}$ m^2 b. i. $x(240 - 3x)$ m^2 ii. 4 800 m^2 when $x = 40$ m
7. 1521 m^2 8. $12\frac{1}{2}$ 9. 6 10. 32 11. 2
12. $4\sqrt{2}$ $x = 1\,000$, $y = \frac{4}{1000}$ 13. $2\sqrt{30}$ 14. $\frac{20}{3}(3)^{\frac{2}{5}} = 10.34$
15. $4\sqrt{10}$ 16. 150 cm^2 17. $S = 48\frac{1}{6}$, $x = 4\frac{1}{4}$, $y = 5\frac{2}{3}$
18. Minimum is -16, maximum is 20 19. Minimum is -16, maximum is144

Chapter 22 – Problems and Investigations

1. Minimum area when circle is of circumference $\frac{L\pi}{4 + \pi}$ and square of length $\frac{4L}{4 + \pi}$.
Maximum area when the string is turned into a circle of radius $\frac{L}{2\pi}$.
2. Maximum area is a square only of side $\frac{L}{4}$.
Minimum area is a square of side $\frac{1}{4}\left(\frac{4L}{4 + 3\sqrt{3}}\right)$ and triangle of side $\frac{1}{3}\left(\frac{3\sqrt{3}\,L}{4 + 3\sqrt{3}}\right)$.
3. The maximum area is obtained with one triangle. Increasing the number of triangles decreases the area.

Chapter 23: Basic Trigonometry

Exercise 23a

1. a. $\frac{4}{5}$ b. $\frac{3}{5}$ c. $\frac{3}{5}$ d. $\frac{4}{5}$ e. $\frac{4}{3}$ f. $\frac{5}{13}$
g. $\frac{3}{4}$ h. $\frac{12}{13}$ i. $\frac{12}{5}$
2. a. 4.1 b. 51.5° c. 4.1 d. 9.2 e. 28.0°
3. N59°W 4. 17.3 km 5. 8.2 cm 6. 77.3 7. 1.8°
8. 6.6 m; 4.6 m; 67.5° 9. 16.4 cm
10. a. 17.5 cm b. 14.2 cm
11. a. 7.6 b. 23.2° 12. a. 45.1° b. 12.0 c. 47.6 d. 16.3
13. a. 32° b. 14.4 c. 9.4 d. 15.3 e. 19.1°
14. a. 40.1° b. 11.0 c. 27.9° d. 19.2 e. 34.7° f. 34.7°
15. a. 9.7 b. 44.1° c. 59.0° d. 7.2
16. 14.3° 17. 5.3 m 18. 19.6 cm 19. 17.9 m 20. 4.2 km

Exercise 23b

1. cosine = $\frac{\sqrt{8}}{3}$, tangent = $\frac{1}{\sqrt{8}}$ 2. sine = $\frac{\sqrt{21}}{5}$, tangent = $\frac{\sqrt{21}}{2}$
3. i. $\frac{\sqrt{15}}{4}$ ii. $\frac{\sqrt{15}}{4}$ iii. $\frac{1}{\sqrt{15}}$ iv. $\sqrt{15}$ v. 1
4. $\frac{5}{4}$, $\frac{5\sqrt{15}}{4}$ 5. 9, $3\sqrt{10}$ 6. $\frac{5}{\sqrt{3}}$, $\frac{10}{\sqrt{3}}$
7. i. $\frac{2}{\sqrt{3}}$ ii. $\frac{2\sqrt{2}}{\sqrt{3}}$ iii. $\frac{\sqrt{2}}{\sqrt{3}}$ iv. $2\sqrt{2}$
8. i. $\sqrt{2}$ ii. $\sqrt{2}$ iii. $\frac{1}{\sqrt{2}}$ iv. $\frac{\sqrt{2}}{\sqrt{3}}$

Chapter 23 – Problems and Investigations

1. a, b, c, f, g, h, i, are true 2. $\frac{8R^3}{3\sqrt{3}}$

Chapter 24: Cosine, Sine and Area Rules

Exercise 24a

1. a. 5.0 b. 60° 2. a. 12.4 b. 51.0°

3. a. 6.0 b. 60.5 4. a. 78.54 b. 1.9
5. a. 328 m b. 79°

Exercise 24b

1. a. 7.3 b. 9.7 c. 25
2. a. 6.0 b. 8.7
3. a. 22.4 b. 14.4 c. 233.2
4. a. 187.86 b. E48.8°N
5. 162 cm
6. a. 3.5 b. 7.3 cm², 374 cm³
7. a. $1 114 b. 1 282.6 m²
8. a. 5.27 b. 5.44
9. a. W46.2°N b. 15.07 km
10. $\frac{2}{\sqrt{3}}$

Chapter 24 – Problems and Investigations

1. d, g, h, j, are true
2. a. true b. true

Chapter 25: Sine, Cosine and Tangent Functions

Exercise 25a

1. a. -cos 57° b. sin 44° c. -tan 35° d. -tan 7° e. cos 50° f. -sin 76°
g. tan 34° h. -sin 18° i. -cos 18° j. sin 40° k. cos 32° l. -tan 20°
m. cos 46° n. sin 47° o. cos 53°

2. a. $\frac{1}{\sqrt{2}}$ b. $\frac{\sqrt{3}}{2}$ c. $\frac{1}{\sqrt{3}}$ d. $-\frac{\sqrt{3}}{2}$ e. $-\frac{1}{2}$ f. $\sqrt{3}$
g. $-\frac{1}{\sqrt{3}}$ h. -1 i. $\frac{\sqrt{3}}{2}$ j. $\frac{1}{\sqrt{2}}$ k. $-\frac{1}{\sqrt{2}}$ l. $-\frac{1}{\sqrt{2}}$
m. $-\frac{1}{\sqrt{2}}$ n. $\frac{\sqrt{3}}{2}$ o. $-\frac{\sqrt{3}}{2}$

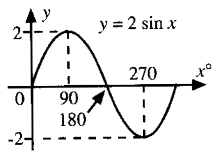
Exercise 25b

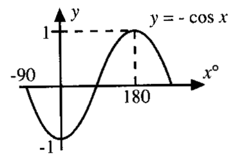
1. 18.5°, 161.5°
2. 64.4°, 295.6°
3. 129.6°, 309.6°
4. 117.5°, 242.5°
5. 300.4°, 239.6°
6. 34.3°, 214.3°
7. 25.7°, 86.3°
8. 188.6°, 277.4°
9. 62.5°, 191.5°
10. 124.2°, 149.8°

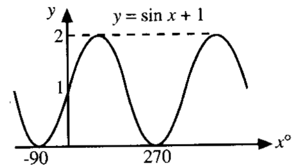
Exercise 25c

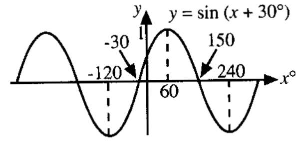
1. a. 11.4, 48.6, 131.4, 168.6, 251.4, 288.6 b. 38.8, 141.2, 218.8, 321.2
c. 17.7, 77.7, 137.7, 197.7, 257.7, 317.7 d. 73.8, 163.8, 253.8, 343.8
e. 31.1, 121.1, 211.1, 301.1, 58.9, 148.9, 238.9, 328.9
f. 104.2, 284.2, 165.8, 345.8 g. 18.7, 108.7, 198.7, 288.7
h. 14.9, 104.9, 194.9, 284.9, 30.1, 120.1, 210.1, 300.1 i. 15.4, 344.6 j. 196.4
2. a. {71.6°, 288.4°} b. {28.9°, 151.1°} c. {234.4°, 305.6°}
d. {115.1°, 295.1°} e. {82.2°, 277.8°} f. {66.5°, 246.5°, 426.5°}
g. {22.9°, 97.1°, 142.9°} h. {3.9°} i. {24.0°}
j. {9.2°, 99.2°, 189.2°} k. {169.8°, 330.2°} l. {3.1°, 56.9°, 123.1°, 176.9°}
3. a. {225°, 315°} b. {60°, 120°} c. {150°, 330°}
d. {180°} e. {45°, 315°} f. {60°, 240°}
g. {210°, 330°} h. {30°, 150°, 270°} i. {15°, 75°, 195°, 255°}
j. {15°, 165°, 195°, 345°} k. {0°, 60°} l. {150°, 270°}
m. {90°, 180°} n. {30°, 120°, 210°, 300°} o. {45°, 135°, 225°, 315°}

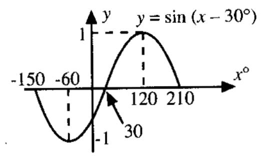
Exercise 25d

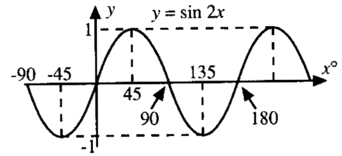
1. a.


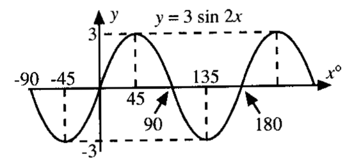
b.


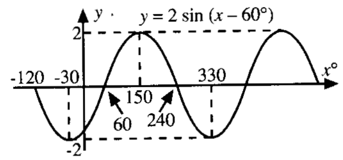
c.


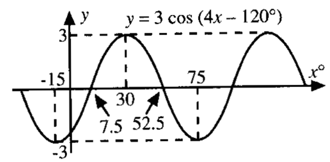
d.


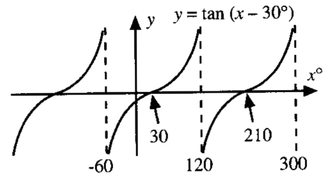
e.


f.


g.


h.


i.


j.


2. i. $y = -3 \sin x$ ii. $y = 4 \cos (x - 20°)$
iii. $y = 5 \cos 2x$ iv. $y = 0.5 \cos (2x + 45°)$
3. e. or $y = \sin (90° - x)$

Exercise 25e

1. a. h = 0.94 sin (30t°) b. h = -0.94 cos (30t°)
2. d = 5.4 + 1.4 cos (30t°)
3. h = 5 sin (72t°)
4. a. h = 6 sin (540t°), where t is in minutes b. After $3\frac{1}{3}$ seconds
5. About $3\frac{1}{2}$ hours

Chapter 25 – Problems and Investigations

1. 2 sin ((n – 1)90°)
2. 3 + 5 sin ((n – 1)90°)
3. n + 2 sin ((n – 1)90°)
4. 3 + 9 sin² ((n – 1)90°)
5. 2n sin ((n – 1)90°)

Chapter 26: Radian Measure

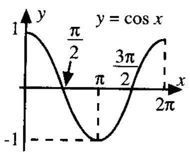
Exercise 26a

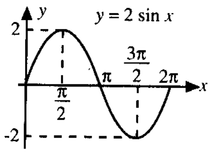
1. a. 0.314 b. 0.663 c. 2.721 d. 4.309 e. 7.292
2. a. 149° b. 195° c. 67.6° d. 32.7° e. 1.72°
3. a. 120° b. 90° c. 10 d. 67.5° e. 144°
   f. 105° g. 75° h. 22. i. 540° j. 150°
4. a. $\frac{2\pi}{3}$ b. $\frac{3\pi}{4}$ c. $\frac{7\pi}{4}$ d. $\frac{9\pi}{4}$ e. $\frac{\pi}{8}$
   f. $\frac{2\pi}{5}$ g. $\frac{3\pi}{5}$ h. $\frac{7\pi}{15}$ i. $\frac{11\pi}{15}$ j. $\frac{3\pi}{20}$
5. 11π or 34.54
6. 6
7. 2.39 per second; 143.2 per minute
8. 24π or 75.4
9. $\frac{20\pi}{3}$ or 20.94
10. $2\frac{1}{3}$
11. a. $\frac{20\pi}{3}$ or 20.94 b. $\frac{32}{\pi}$ laps (approx. 10.2 laps)
12. Approximately 398
13. a. 360 rpm b. 8π or 25 rad per sec.
14. a. 60π or 188.5 rads per sec. b. 9.9 ms^{-1}
15. a. 108 rpm b. $8\frac{1}{3}$ cm

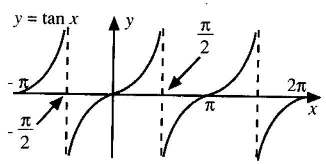
Exercise 26b

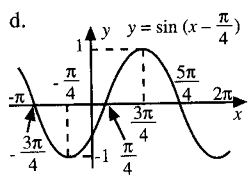
1. a. 1 b. $-\frac{1}{\sqrt{2}}$ c. $\frac{1}{\sqrt{2}}$ d. -1 e. $-\sqrt{3}$ f. $-\frac{1}{2}$
   g. $-\frac{1}{2}$ h. 1 i. $\frac{\sqrt{3}}{2}$
2. d
3. c
4. b
5. d
6. b
7. a. $\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ b. $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$ c. $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ d. $\left\{-\frac{2\pi}{3}, \frac{\pi}{3}\right\}$
   e. $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ f. $\left\{-\frac{\pi}{3}, -\frac{2\pi}{3}\right\}$ g. $\left\{-\frac{\pi}{4}, \frac{\pi}{4}\right\}$ h. $\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}$
   i. $\left\{\frac{2\pi}{3}, 0\right\}$ j. $\left\{\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}\right\}$ k. $\left\{\frac{\pi}{4}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \frac{7\pi}{4}\right\}$
   l. $\left\{\frac{3\pi}{2}\right\}$ m. $\left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}\right\}$ n. $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$ o. No solution
8. a. {0.41, 2.73} b. {1.27, 5.02} c. {0.67, 3.82} d. {2.55, 5.70}
   e. {0.54, 3.68} f. {2.68, 5.82} g. {0.34, 2.8} h. {2.42, 3.86}
   i. {1.07, 2.08} j. {2.16, 5.3} k. {2.29, 5.43} l. {1.72, 4.56}

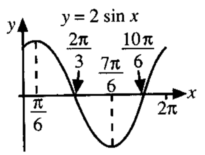
Exercise 26c

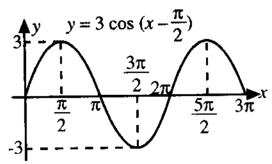
1. a.


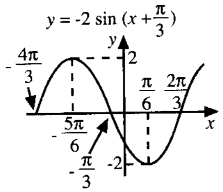
b.


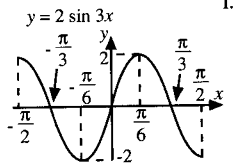
c.


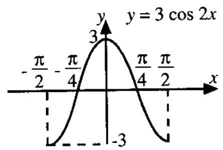
d.


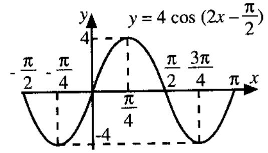
e.


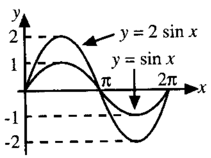
f.


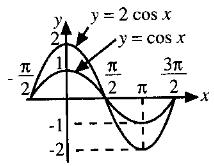
g.


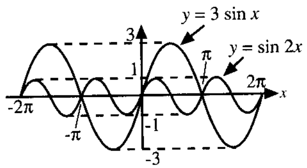
h.


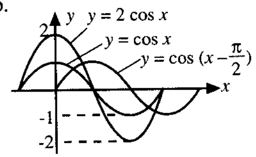
i.


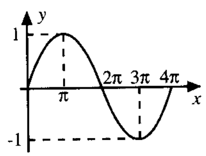
j.


2. a. ii. or $y = \sin 2x$
   b.




3. a. $f: x \rightarrow \sin x$ b. $f: x \rightarrow -\sin x$ c. $f: x \rightarrow -\sin(\frac{\pi}{2} - x)$
   d. $f: x \rightarrow -\sin(\frac{\pi}{2} - x)$ e. $f: x \rightarrow -\sin(\frac{\pi}{2} - x)$
4. a. $g(x) = \cos 2x$, $h(x) = -\sin x$ b. 2π c. $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
5. a.


b.


c. $m(x) = \sin x$
6. a. $(\pi, -2)$ b. $(\frac{\pi}{2}, -3)$ c. $(\frac{\pi}{3}, -1)$ d. $(\frac{\pi}{3}, -4)$ e. $(\frac{\pi}{4}, -2)$
7. a. $y = \sin x$ b. $y = \cos x$ c. $y = -\cos x$ d. $y = -\sin x$ e. $y = 4 \sin x$
8. a. 11.5 m b. After 6 seconds when the wheel is 15 m up.
9. a.


b. 4π c. $\frac{1}{2}$ d. $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$

10. a. $y = 2\cos 2x$ b. $y = -\cos 2x$ c. $y = 2\sin 3x$ d. $y = \cos\frac{x}{2}$

Chapter 26 – Problems and Investigations

1. Usually $\frac{n}{\pi} + 2$ interactions, occasionally $\frac{n}{\pi} + 1$
2. Consistent with h = 8 + 3 sin (180t), where angles are in degrees
3. h = 6 + 1.5 sin (60t), where angles are in degrees

Chapter 27: Arc Length and Sector Area

Exercise 27a

1. a. S = 3m; A = 7.5m² b. S = 8.4 m; A = 29.4 m² c. S = 56.5 m; A = 339.3 m²
   d. S = 9.9 m; A = 31.2 m² e. S = 32.5 m; A = 118.6 m²
2. a. S = 2.1; A = 3.1 b. S = 22.9; A = 108.7 c. S = 37.7; A = 150.8
   d. S = 11.7; A = 49.3 e. S = 10.1; A = 32.2
3. a. $\frac{5}{7}$ rad or 0.71 rad b. 38.2° c. $1\frac{1}{3}$ rad or 76.4°
   d. 3.5 m e. 7.0 cm²
4. a. 42.1° b. 39 cm² c. 9.8 cm d. 9.5 cm e. 300 cm²

Exercise 27b

1. 3.82 cm² 2. a. 0.85 b. 6.6 3. a. 1.21 m² b. 5.05 m
4. $(4\pi - 6\sqrt{3})$ m² (approx. 2.17 m²) 5. $(9 - \frac{9\pi}{4})$ m² (approx. 1.93 m²) 6. 13.8 m²

Chapter 27 – Problems and Investigations

1. Maximum occurs when radius is $2\frac{1}{2}$ 2. $\frac{n}{2}\sin\frac{2\pi}{n}$ 3. Not possible, unless θ = 0

Chapter 28: Antidifferentiation

Exercise 28

1. a. $\frac{x^2}{2} + c$ b. $\frac{x^4}{4} + c$ c. $\frac{2x^5}{5} + c$
   d. $\frac{x^{12}}{12} + \frac{x^{13}}{13} + c$ e. $\frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} + c$ f. $\frac{x^8}{4} + \frac{3x^7}{7} + c$
   g. $x^4 - \frac{x^6}{3} + c$ h. $\frac{x^4}{2} - \frac{3x^5}{5} + c$ i. $\frac{x^3}{3} - \frac{3x^2}{2} + c$
   j. $2x + c$ k. $3x + c$ l. $\frac{x^2}{2} - 3x + c$
   m. $\frac{x^3}{3} - \frac{3x^2}{2} + 4x + c$ n. $\frac{x^3}{3} + \frac{7x^2}{2} + 12x + c$ o. $\frac{4x^3}{3} + 6x^2 + 9x + c$
   p. $\frac{mx^3}{3} + c$ q. $\frac{3}{2}x^4 + 3x^3 + 4x^2 + 12x + c$ r. $\frac{4x^5}{5} + \frac{20x^3}{3} + 25x + c$
   s. $a^2x + c$ t. $\frac{x^2}{2b} + c$ u. $\frac{cx^2}{2} + \frac{x^3}{3d} + k$

   v. $\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx + D$ w. $\frac{x^3}{3A} + \frac{x^2}{2B} + \frac{x}{C} + D$
   x. $\frac{A^2x^5}{5} + \frac{ABx^4}{2} + \frac{(B^2 + 2AC)x^3}{3} + BCx^2 + C^2x + D$ y. $\frac{x^5}{5} + \frac{Ax^4}{2} + \frac{A^2x^3}{3}$
2. a. $\frac{3x^2}{2} + c$ b. $\frac{3x^5}{5} + c$ c. $\frac{3x^8}{8} + c$ d. $\frac{2x^9}{9} + c$
   e. $\frac{x^{14}}{14} - \frac{x^{11}}{11} + c$ f. $\frac{x^8}{4} + \frac{4x^6}{3} + c$ g. $\frac{x^5}{5} - x^4 + \frac{3x^2}{2} + c$
   h. $\frac{x^4}{4} + x^3 + \frac{x^2}{2} + 3x + c$ i. $\frac{4x^3}{3} + 10x^2 + 25x + c$ j. $\frac{Ex^2}{2} + Dx + C$
3. a. $2x^2 + c$ b. $\frac{x^3}{9} + c$ c. $\frac{7x^4}{4} + c$ d. $r^2 + c$ e. $\frac{8r^9}{9} + c$
   f. $\frac{5t^7}{7} + c$ g. $8x + c$ h. $ax + c$ i. $t^5x + c$ j. $\frac{t^6}{3} + c$
   k. $x^2 + 4x + c$ l. $\frac{x^4}{4} - x^3 + \frac{5x^2}{2} + 2x + c$ m. $\frac{16x^3}{3} + 12x^2 + 9x + c$
   n. $\frac{4x^5}{5} + \frac{4x^3}{3} + x + c$ o. $\frac{x^4}{4} + x^3 + c$
4. a. $-\frac{1}{2}x^{-2} + c$ b. $-\frac{4}{5}x^{-5} + c$ c. $-\frac{1}{6}x^{-6} + c$ d. $-\frac{1}{9}x^{-9} + c$
   e. $-\frac{3}{11}x^{-11} + c$ f. $-u^{-5} + c$ g. $-\frac{1}{2}v^{-2} + c$ h. $-\frac{3}{4}x^{-4} + c$
   i. $-\frac{1}{8}x^{-2} + c$ j. $-\frac{1}{20}x^{-4} + c$ k. $-\frac{1}{14}u^{-2} + c$ l. $-\frac{1}{27}x^{-6} + c$
   m. $-\frac{1}{5}x^{-3} + c$ n. $-\frac{1}{3}x^{-3} - x^{-2} - x^{-1} + c$ o. $-4u^{-1} + \frac{2}{3}u^{-2} - \frac{1}{27}u^{-3} + c$
5. a. $\frac{2}{3}(x)^{\frac{3}{2}} + c$ b. $\frac{4}{5}(x)^{\frac{5}{4}} + c$ c. $\frac{3}{7}(x)^{\frac{7}{3}} + c$ d. $\frac{5}{12}(t)^{\frac{12}{5}} + c$
   e. $\frac{8}{15}(u)^{\frac{15}{8}} + c$ f. $\frac{8}{11}(t)^{\frac{11}{8}} + c$ g. $3(x)^{\frac{5}{3}} + c$ h. $\frac{27}{7}(x)^{\frac{7}{3}} + c$
   i. $\frac{40}{11}(x)^{\frac{11}{4}} + c$ j. $\frac{x^3}{5} + c$ k. $\frac{7}{15}(x)^{\frac{5}{2}} + c$ l. $2(x)^{\frac{1}{2}} + c$
   m. $-2(x)^{-\frac{1}{2}} + c$ n. $-\frac{3}{2}(x)^{-\frac{2}{3}} + c$ o. $5(x)^{\frac{1}{5}} + c$ p. $-12(u)^{-\frac{1}{4}} + c$
   q. $\frac{35}{24}(x)^{\frac{4}{7}} + c$ r. $\frac{x^2}{2} + 3x + c$ s. $\frac{3}{5}(x)^{\frac{5}{3}} + \frac{3}{2}(x)^{\frac{4}{3}} + x + c$ t. $2u^2 - 12u^{-1} - \frac{9}{4}u^{-4} + c$

Chapter 29: Finding the Constant of Integration

Exercise 29a

1. $\frac{x^2}{2}$ 2. $\frac{x^3}{3} + \frac{x^2}{2} + 2$ 3. $x^3 + 3$ 4. $x^4 + x^2 - 15$
5. $\frac{x^4}{2} + x - 3$ 6. $\frac{3x^2}{2} + 2x - 15\frac{1}{2}$ 7. $\frac{5x^3}{3} - 95\frac{2}{3}$ 8. $\frac{2x^3}{3} - \frac{x^2}{2} + 3x - 128$

9. $\frac{3x^2}{2} - \frac{x^3}{3} + 402$ 10. $\frac{2x^3}{3} - \frac{3x^2}{2} + 2x + 3\frac{5}{6}$ 11. $\frac{x^2}{2} + 3x - 7$ 12. $21 - r^2$

13. a. $t^2 + t + 12$ b. 13.6 seconds 14. a. $11\,000 - 60t - t^2$ b. After 79.1 seconds

15. a. $149 - 5x + \frac{6}{x}$ b. After 9.92 km

Exercise 29b

1. a. 80 m b. 120 m c. 80 m d. After 10 minutes
   e. $50 - 10t$ ms^{-1} f. 20 ms^{-1} g. 125 m h. -10 ms^{-2} i. -10 ms^{-2}
2. a. 40 m b. 25 m c. 6 seconds d. 30 - 10t ms^{-1}
   e. -10 ms^{-1} f. -30 ms^{-1} g. 45 m h. -10 ms^{-2}
3. a. 80 ms^{-1} b. After 25 seconds c. $(100t - 2t^2 + 100)$ m
   d. 382 m e. -4 ms^{-2}
4. a. $(50t - \frac{3t^2}{2} + 10)$ cm b. 186 cm c. 256 cm
   d. After $16\frac{2}{3}$ seconds e. -3 cms^{-1}
5. a. $(5 + 3t)$ cms^{-1} b. 35 cms^{-1} c. After $13\frac{1}{3}$ seconds
6. a. $6t - 3t^2$ b. $6 - 6t$ c. 0. d. 0 e. When t = 0 or 3
7. a. $\frac{2}{3}$ m b. -3 ms^{-1} c. After 2 seconds d. $1\frac{1}{3}$ m
8. a. 40 ms^{-1} b. -10 ms^{-2} d. 4 seconds e. 80 m f. 8 seconds
9. a. 3.8 m b. $-\frac{1}{10}$ ms^{-2} c. 40 seconds 10. $10\frac{2}{3}$ m

Chapter 29 – Problems and Investigations

1. About 90 metres 2. 3.3 ms^{-2}

Chapter 30: Definite Integrals and Area

Exercise 30a

1. a. 6 b. $41\frac{1}{3}$ c. $7\frac{1}{2}$ d. $4\frac{4}{15}$ e. $1\frac{1}{3}$ f. 2
   g. 18 h. 9 i. $7\frac{1}{2}$ j. 21 k. $-42\frac{1}{12}$ l. $5\frac{13}{24}$
   m. $\frac{5}{16}$ n. $-8\frac{1}{3}$ o. 3.54
2. a. $12\frac{2}{3}$ b. $48\frac{3}{4}$ c. 12.4 d. $\frac{1}{6}$ e. $\frac{8}{225}$ f. $\frac{5}{9}$
   g. 1.32 h. 0.701 i. 0.054 j. 0.694

Exercise 30b

1. a. $12\frac{2}{3}$ b. $4\frac{2}{3}$ c. $36\frac{2}{3}$ 2. 6 3. 10 4. 39
5. $1\frac{2}{3}$ 6. $10\frac{2}{3}$ 7. $20\frac{5}{6}$ 8. $4\frac{1}{4}$ 9. 2 10. $7\frac{2}{3}$

Exercise 30c

1. $4\frac{1}{2}$ 2. 36 3. $4\frac{1}{2}$ 4. $\frac{32}{3}$ 5. $21\frac{1}{3}$

Exercise 30d

1. a. 4 b. $6\frac{1}{2}$ c. $-\frac{1}{2}$ d. $-3\frac{1}{2}$ e. $-3\frac{1}{2}$ f. -2
   g. 3 h. 1 i. $-4\frac{1}{2}$ j. 0
2. a. 1 b. 3 c. 4 d. 7 e. 3 f. $5\frac{1}{2}$
   g. $-1\frac{1}{2}$ h. 4 i. -3 j. $-\frac{2}{5}$ (Answers approximate)

Chapter 30 – Problems and Investigations

1. a. $4\frac{2}{3}$ b. 5 2. a. 18 b. 17.5 3. $\frac{1}{3}$ 4. $\frac{4a}{3}$
5. 1.31 6. $-\frac{2}{3}$

Chapter 31: Statistics

Exercise 31b

1. Dates of sports days 2. Soap brands 3. Subjects 4. Ages of mice
5. Colours of houses 6. Attitudes to politicians 7. Distances travelled
8. Toys 9. Attitudes to government policy 10. Number of passengers

Chapter 33: Sampling

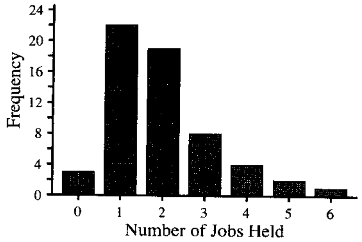
1. a. Selected so that every member of the population has an equal chance of selection.
   b. i. and ii. are methods ensuring random selection.
2. a. The probability of getting a head is different to the chance of getting a tail.
   b. By experiment. c. i, v
3. a, c, d, e, f, g, h, i, will tend to produce biased results.
4. a. People not on the electoral roll are excluded. Unlikely to affect validity.
   b. This is all right only if the views of people are similar from suburb to suburb.
   c. Not very satisfactory. The proportions of people in the factory with different views are unlikely to be the same as the population as a whole.
   d. This method is all right only if the views of people in the street reflect those of the population.
   e. Definitely not satisfactory. Teachers are a special interest group whose views on education will tend to be different from the population as a whole.
5. a. 625 members of A, 325 members of B, 1 050 members of C
   b. 39 of B and 126 of C c. 200
6. Oliver, Sebastian, John, Gareth, Harry 7. Edward, Winsome, Elaine, Sebastian
8. 8 males, 12 females 9. 6 females
10. {Edward, Nicholas, Winston, Frank, Xenia, Tammy, Hilda, Bronwyn, Lynette, Jean}
11. {Paul, Adam, Harry, Ramesh, Luke, Darius, Danielle, Jasmin, Tracey, Rosalie, Hilda, Valerie, Jean, Winsome, Helen}
12. {Conan, Edward, Gwyneth, Holly, Jessica, Luke, Nicholas, Ben, Danielle, Fleur}

13. {Rosalie, Valerie, Helen, Odette, Jean}
14. There are many such clusters.
15. A sample of six chosen by starting with the eighth name, then selecting every third subsequent name.

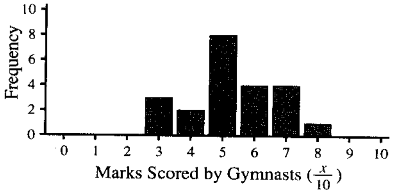
Chapter 34: Basic Data Display

Exercise 34a

1.



2.



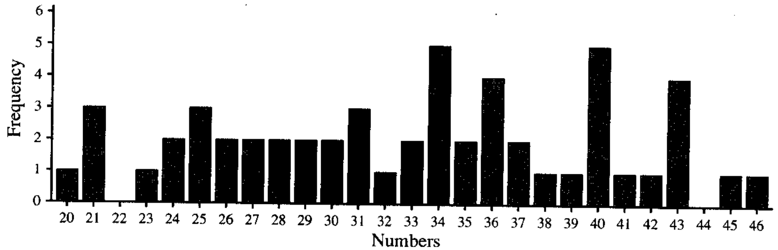
Exercise 34b

1. a.

Number	Tallies	Frequency
20	|	1
21	|||	3
22		0
23	|	1
24	||	2
25	|||	3
26	||	2
27	||	2
28	||	2
29	||	2
30	||	2
31	|||	3
32	|	1
33	||	2

Number	Tallies	Frequency
34	𝍸	5
35	||	2
36	||||	4
37	||	2
38	|	1
39	|	1
40	𝍸	5
41	|	1
42	|	1
43	||||	4
44		0
45	|	1
46	|	1

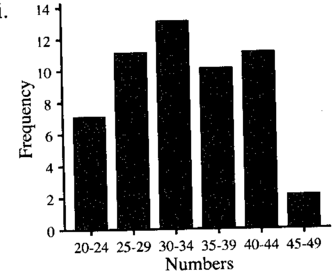
b.



c.

Number	Tallies	Frequency
20-24	𝍸 ||	7
25-29	𝍸 𝍸 |	11
30-34	𝍸 𝍸 |||	13
35-39	𝍸 𝍸	10
40-44	𝍸 𝍸 |	11
45-49	||	2
		54

i.

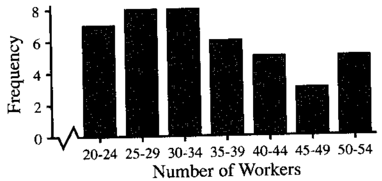


ii. The pattern of the above graph demonstrates that the frequency is small at the two ends, while the majority of the numbers are situated in the middle region where the frequency peaks.

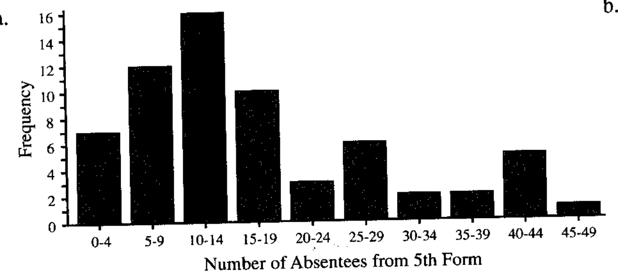
iii. The most suitable graph is the second graph, with the data grouped in fives. This is because we get a general idea on the spread of the numbers by using groups of five. Using the total number of each **individual** number does not reveal any **major** trends.

2.

Number of Workers	Tallies	Frequency
20-24	𝍸 ||	7
25-29	𝍸 |||	8
30-34	𝍸 |||	8
35-39	𝍸 |	6
40-44	𝍸	5
45-49	|||	3
50-54	𝍸	5
		42



3. a.



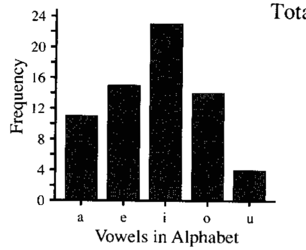
b. i. 19 days
ii. 19 days
iii. 35 days
iv. 3 days or less
v. $16 \leq$ days ≤ 19
vi. 16 days

c. Since the data is arranged in groups of four, it is difficult to answer questions which concern ungrouped data, e.g. v.

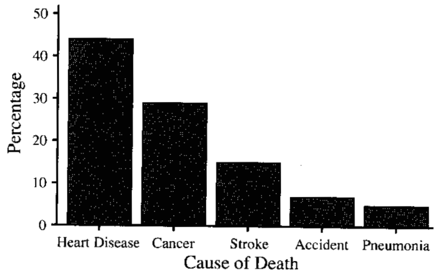
Exercise 34c

1.

Vowel	Tallies	Frequency	Relative Frequency
a	𝍸 𝍸 I	11	11 ÷ 67 = 0.164
e	𝍸 𝍸 𝍸	15	15 ÷ 67 = 0.224
i	𝍸 𝍸 𝍸 𝍸 II	23	23 ÷ 67 = 0.343
o	𝍸 𝍸 IIII	14	14 ÷ 67 = 0.209
u	IIII	4	4 ÷ 67 = 0.060
	Totals	67	1.000



2.

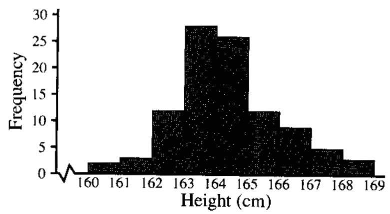


44% Heart Disease
29% Cancer
15% Stroke
7% Accident
5% Pneumonia

Exercise 34d

1. a. Discrete b. Continuous
   c. Discrete d. Continuous
   e. Discrete f. Discrete
   g. Discrete h. Continuous
   i. Discrete j. Discrete

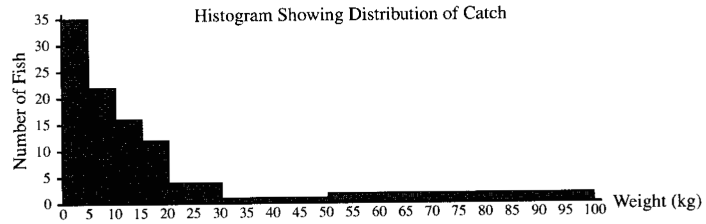
2.



3. a. 60 b. 205
   c. 205 d. $\frac{182}{252}$
   e. $\frac{175}{252}$ f. $\frac{152}{252}$
   g. $\frac{176}{252}$ h. $\frac{101}{252}$
   i. 54% j. 23%

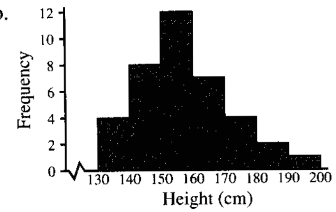
Exercise 34e

1.



2. a. i. No labels on axes
   ii. No title
   iii. Incorrect scale on horizontal axis
   iv. Incorrect values drawn for histogram

   b.



Chapter 35: Central Tendency and Spread

Exercise 35a

1. a. 10.125 b. $77\frac{1}{7}$ c. 4.79 (2 dp)
2. 80
3. There are many such sets, eg 21, 22, 23, 24, 25.

Exercise 35b

1. a. Median = $27\frac{1}{2}$ LQ = 16 UQ = 42 There is no mode
   b. Median = 4 LQ = 3.12 UQ = 4.3 Mode = 4.3
   c. Median = 3 LQ = 2 UQ = 3 Mode = 3
   d. Median = $48\frac{1}{2}$ LQ = 32 UQ = 73 There is no mode
2. There are many such sets

Exercise 35c

1. a. Median 2.5, mode 2, range 10
2. a. Median 4, mode 4, range 7
   c. 2.3 d. larger
3. a. 1.41 b. 1.41
4. There are many of each

Exercise 35d

1. a. $\bar{x} = 5, s = 3$
   b. i. $\bar{x} = 15, s = 3$ ii. Mean is increased by 10, standard deviation is unchanged.
   c. i. $\bar{x} = 50, s = 30$ ii. Both mean and standard deviation are multiplied by 10.
2. a. $\bar{x} = 11, s = 2$ b. $\bar{x} = 30, s = 10$ c. $\bar{x} = 26, s = 2$ d. $\bar{x} = 120, s = 40$
   e. $\bar{x} = 4, s = 2$ f. $\bar{x} = 3, s = 1$ g. $\bar{x} = 6, s = 2$

3. a. Median = 4, mode = 4, range = 6, mean = 4, s = 2
   b. i. Median = 7, mode = 7, range = 6, mean = 7, s = 2
      ii. Median = 12, mode = 12, range = 18, mean = 12, s = 6
      iii. Median = 104, mode = 104, range = 6, mean = 104, s = 2
      iv. Median = 400, mode = 400, range = 600, mean = 400, s = 200
      v. Median = 2, mode = 2, range = 6, mean = 2, s = 2
      vi. Median = 2, mode = 2, range = 3, mean = 2, s = 1
4. a. i.

Number	1	2	3	4	5	Total
Frequency	10	10	10	10	10	50

   ii. mean = 3, standard deviation = $\sqrt{2}$ or 1.4142 (4 dp)
   b. Smaller; values are clustered closer to the mean.
   c. Mean increases by 10, standard deviation unchanged.

Exercise 35e

1. a. $\bar{x}$ = 11.5, s = 7.9 b. $\bar{x}$ = 4.9, s = 5.2 2. a. $\bar{x}$ = 65.1, s = 30.2 b. $\bar{x}$ =57.4, s = 41.2
3. a. First has mean of about 2.3 the second has mean of about 4.7, standard deviation is about 1.5 for both.
   b. i. For first, $\bar{x}$ 3, s = 1.5 For second, $\bar{x}$ = 5.7, s = 1.5
      ii. For first, $\bar{x}$ = 4.6, s = 3.0 For second, $\bar{x}$ = 9.4, s = 3.0
      iii. For first, $\bar{x}$ = 2.3, s = 1.5 For second, $\bar{x}$ = 4.7, s = 1.5

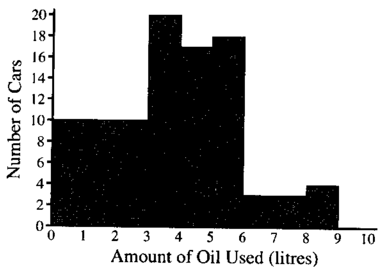
Chapter 36: Other Data Displays

Exercise 36a

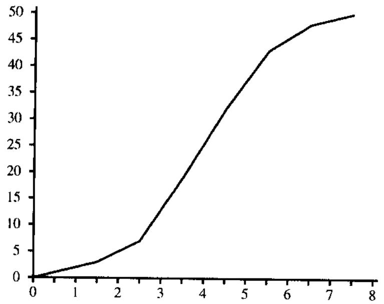
1. a. 50% b. 35% c. 60% d. 4 litres e. 2.5 litres

   f.

Amount of Oil Used (L)	No. of Cars
$0 \le x < 1$	10
1 < 2	10
2 < 3	10
3 < 4	20
4 < 5	17
5 < 6	18
6 < 7	3
7 < 8	3
8 < 9	4
9 < 10	0

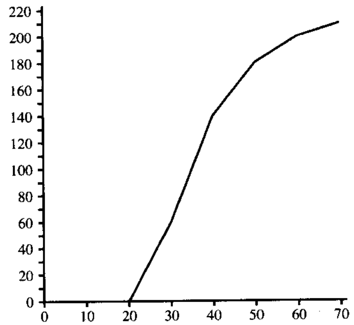


2. a.



   b. about 4 years
   c. about 4.8 years
   d. about 76%
   e. about 2.1 years

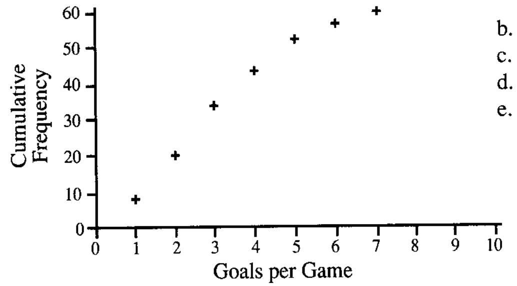
3. a.



   b. 135 ± 5
   c. 35.5 ± 0.5 years
   d. 17 ± 1 years
   e. 80 ± 5

4. a. 20 b. 17 c. 11 d. 3 e. 9
5. a.

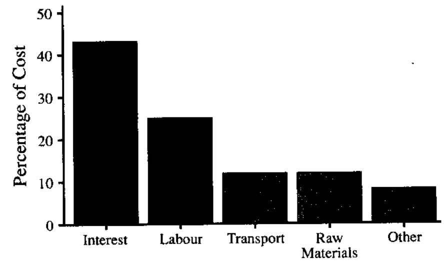
Goals per game	0	1	2	3	4	5	6	7
Cum. Frequency	3	8	20	34	44	52	56	60



   b. Median = 3 goals per game
   c. Mean = 3.38
   d. UQ = 5 goals per game
   e. about 57%

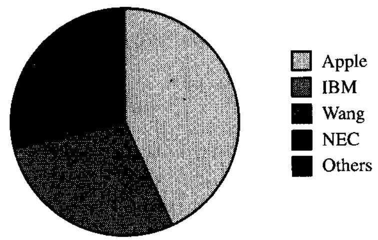
Exercise 36b

1. a. Interest Cost = $27 295 970, Labour Cost = $15 869 750
   b. 43.2° (if exact)
   c.

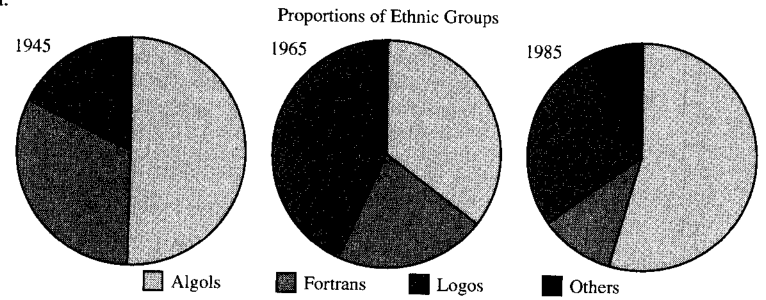


   d. Interest labour etc., are not numerical and cannot be placed on a number line.

2. a.



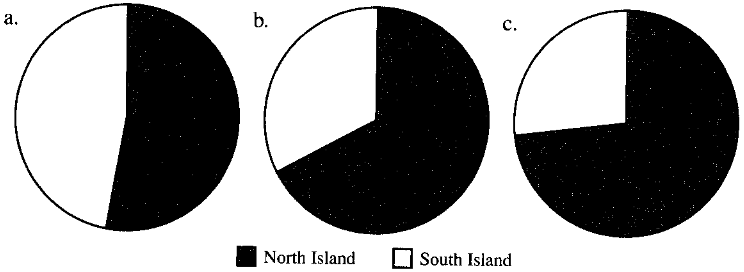
3. a.



b. The group "Fortrans" constantly declined over the thirty years. Conversely, the group "Others" steadily increased while the groups "Algols" and "Logos" both fluctuated.

c. The pie graphs don't show the total number of people represented.

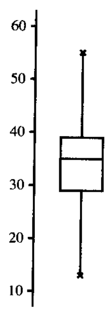
4. a. b. c.



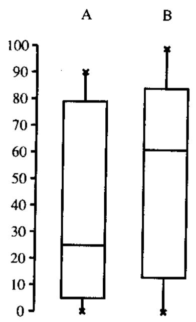
The three graphs illustrate the declining percentage of the NZ population living in the South Island.

Exercise 36c

1.



2. a.



b. Group B is higher in quartiles, median and maximum than Group A. Both have same minimum. Group B has slightly greater spread.

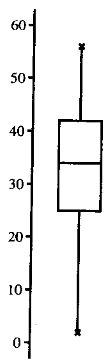
Exercise 36d

1. a.

5	6
4	2 2 4 5
3	1 3 4 4 5 6 7 8
2	2 5 5 6
1	4
0	2

b. Median = 3.4
LQ = 2.5
UQ = 4.2

c.

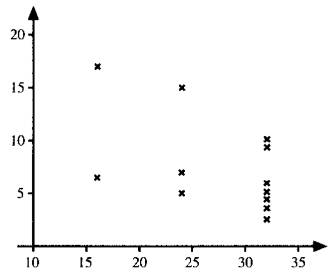


2.

13	0
12	2 4 5
11	1 2 5 5 7 8 9 9
10	8 8 8 9 9 9

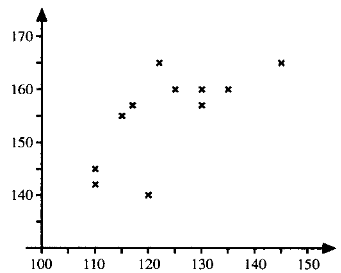
Exercise 36e

1.



Some points have been omitted.
There appears to be no correlation.

2.



There is a tendency for performance in the clean and jerk to rise as the snatch rises.
Some correlation.

Chapter 37: Misleading Uses of Statistics

Exercise 37

1. a. No vertical scale
   b. Suppressed vertical scale
   c. Data missing for 1991, 1993
   d. Uneven vertical scale
   e. Slanting scale
   f. Results for 1995 and 1996 have been combined.
2. a. Slanting scale with Waitakere Products' bars at the far end of the scale creating the illusion that their profits are much smaller than those of Rutherford.
   b. Bars for Rutherford are darker than those of Waitakere which diverts the attention of the observer to those bars. Furthermore, they are fatter, creating the illusion of bigger profits.
   c. Scale for Waitakere is missing. The fact that Waitakere has a diminishing profit against Rutherford's increasing profit is highlighted. Missing scale makes it impossible to tell whether Waitakere's profits are still much larger than those of Rutherford.
3. a. is unfair to Waitakere. Volume of Rutherford's barrel is four times that of Waitakere.
   b. is fair. Volume of Rutherford's barrel is twice that of Waitakere.
   c. is unfair to Rutherford. The volumes of the barrels are the same. All the taps suggest is that Rutherford's oil supplies might run out twice as quickly as Waitakere's.
   d. is unfair to Waitakere. The area of Rutherford's "dollar" is four times that of Waitakere's.
   e. is unfair to Waitakere. The volume of Rutherford's container is about eight times that of Waitakere's.

Chapter 38: Time Series

Exercise 38a

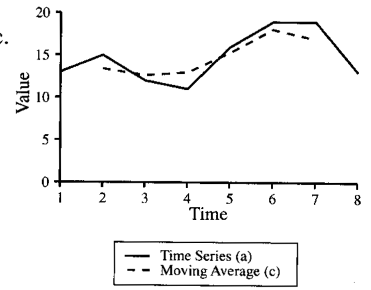
(1) and (2) exhibit a secular upward trend with daily periodic movement. In the case of (1) there is an uncharacteristic "flattening" of this trend for a period of about 10 days near the middle of the graph. In the case of the second graph there is a "jump" at the beginning of June.

Exercise 38b

1. b.

Time	Value	Moving Average
1	13	
2	15	13.333333
3	12	12.666666
4	11	13
5	16	15.333333
6	19	18
7	19	17
8	13	

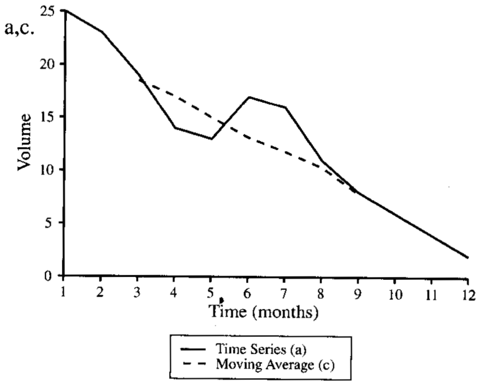
a,c.



2. b.

Time	Volume	Moving Average
1	25	
2	23	
3	19	18.5
4	14	17
5	13	15
6	17	13.1666667
7	16	11.8333333
8	11	10.3333333
9	8	7.8333333
10	6	
11	4	
12	2	

a,c.



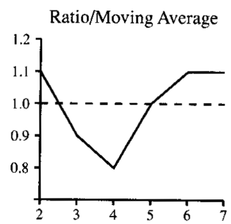
d. 5.3

Exercise 38c

1.

Time	Value	Moving Average	Ratio/Moving Average
1	13		
2	15	13.333333	1.125
3	12	12.666666	0.947368
4	11	13	0.846154
5	16	15.333333	1.043478
6	19	18	1.055555
7	19	17	1.117647
8	13		

Time	Ratio/Moving Average
2	1.1
3	0.9
4	0.8
5	1.0
6	1.1
7	1.1

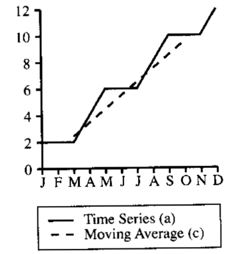


Examination of the graph shows some periodicity of about 20% maximum from the moving average.

2. b.

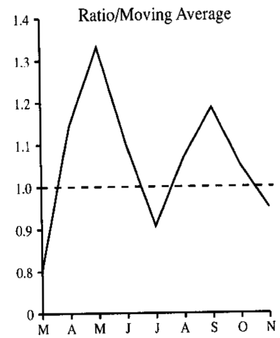
Month	Length	Moving Average	Ratio/Moving Average
Jan	2		
Feb	2		
Mar	2	5	0.8
Apr	4	3.5	1.14
May	6	4.5	1.33
Jun	6	5.5	1.09
Jul	6	6.5	0.92
Aug	8	7.5	1.07
Sep	10	8.5	1.18
Oct	10	9.5	1.05
Nov	10	10.5	0.95
Dec	12		

a,c.



d.

Month	Ratio/Moving Average
Mar	0.8
Apr	1.14
May	1.33
Jun	1.09
Jul	0.92
Aug	1.07
Sep	1.18
Oct	1.05
Nov	0.95



Examining the graph (2d) indicates that there is some periodic fluctuation about the moving average of a maximum of 20%.

3. a = 94, b = 46, c = 62.5, d = 49

Chapter 39: Probability

Exercise 39b

1. a. $\frac{9}{16}$ b. $\frac{7}{16}$ c. 1 d. 0
2. a. $\frac{9}{16}$ b. $\frac{9}{15}$ c. $\frac{7}{16}$ d. $\frac{6}{15}$
3. a. $\frac{8}{20}$ b. $\frac{5}{20}$ c. $\frac{7}{20}$ d. $\frac{12}{20}$
4. a. $\frac{5}{20}$ b. $\frac{5}{19}$ c. $\frac{7}{20}$ d. $\frac{6}{19}$
5. a. $\frac{2}{7}$ b. $\frac{1}{7}$ c. $\frac{3}{7}$ d. $\frac{4}{7}$ e. 1 f. 0
6. a. $\frac{7}{12}$ b. $\frac{1}{12}$ c. $\frac{2}{12}$ d. $\frac{4}{12}$ e. $\frac{2}{12}$ f. $\frac{5}{12}$
7. a. $\frac{4}{9}$ b. $\frac{5}{9}$ c. 1 d. 0 e. $\frac{3}{9}$ f. $\frac{3}{9}$
   g. $\frac{1}{9}$ h. $\frac{5}{9}$
8. a. $\frac{4}{52}$ b. $\frac{13}{52}$ c. $\frac{1}{52}$ d. $\frac{16}{52}$ e. $\frac{36}{52}$ f. $\frac{3}{52}$ g. $\frac{12}{52}$
9. a. $\frac{26}{52}$ b. $\frac{12}{52}$ c. $\frac{6}{52}$ d. $\frac{32}{52}$ e. 0 f. $\frac{20}{52}$ g. $\frac{6}{52}$

10. a. i. $\{B_1, B_2, B_3, W_1, W_2\}$ ii. $\frac{2}{5}$ iii. $\frac{3}{5}$

b. i.

B_1B_1	B_2B_1	B_3B_1	W_1B_1	W_2B_1
B_1B_2	B_2B_2	B_3B_2	W_1B_2	W_2B_2
B_1B_3	B_2B_3	B_3B_3	W_1B_3	W_2B_3
B_1W_1	B_2W_1	B_3W_1	W_1W_1	W_2W_1
B_1W_2	B_2W_2	B_3W_2	W_1W_2	W_2W_2

ii. $\frac{9}{25}$ iii. $\frac{6}{25}$ iv. $\frac{12}{25}$

c. i.

B_1B_2	B_2B_1	B_3B_1	W_1B_1	W_2B_1
B_1B_3	B_2B_3	B_3B_2	W_1B_2	W_2B_2
B_1W_1	B_2W_1	B_3W_1	W_1B_3	W_2B_3
B_1W_2	B_2W_2	B_3W_2	W_1W_2	W_2W_1

ii. $\frac{2}{20}$ iii. $\frac{6}{20}$ iv. $\frac{14}{20}$ v. $\frac{12}{20}$

11. a. i. $\{B_1, B_2, R_1, R_2, W\}$ ii. $\frac{1}{5}$ iii. $\frac{3}{5}$

b. i.

B_1B_1	B_2B_1	R_1B_1	R_2B_1	WB_1
B_1B_2	B_2B_2	R_1B_2	R_2B_2	WB_2
B_1R_1	B_2R_1	R_1R_1	R_2R_1	WR_1
B_1R_2	B_2R_2	R_1R_2	R_2R_2	WR_2
B_1W	B_2W	R_1W	R_2W	WW

ii. $\frac{4}{25}$ iii. $\frac{9}{25}$ iv. $\frac{2}{5}$

c. i.

B_1B_2	B_2B_1	R_1B_1	R_2B_1	WB_1
B_1R_1	B_2R_1	R_1B_2	R_2B_2	WB_2
B_1R_2	B_2R_2	R_1R_2	R_2R_1	WR_1
B_1W	B_2W	R_1W	R_2W	WR_2

ii. $\frac{4}{20}$ iii. $\frac{4}{20}$ iv. $\frac{8}{20}$ v. $\frac{12}{20}$

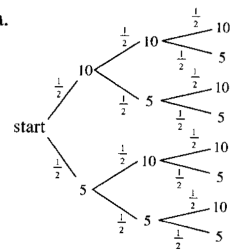
12. a. i. $\{M_1, M_2, M_3, M_4, W_1, W_2\}$ ii. $\frac{1}{3}$ iii. $\frac{1}{3}$ b. i. $\frac{4}{9}$ ii. $\frac{8}{9}$ iii. $\frac{2}{3}$

c. i. $\frac{2}{5}$ ii. $\frac{2}{5}$ iii. $\frac{1}{3}$ iv. $\frac{1}{15}$

Exercise 39c

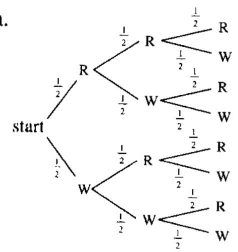
1. a. i. $\frac{4}{9}$ ii. $\frac{2}{9}$ iii. $\frac{5}{9}$ b. i. $\frac{1}{2}$ ii. $\frac{1}{12}$ iii. $\frac{1}{2}$

2. a.



b. i. $\frac{1}{8}$ ii. $\frac{3}{8}$ iii. $\frac{7}{8}$ iv. $\frac{2}{8}$ v. $\frac{7}{8}$

3. a.



b. i. $\frac{1}{8}$ ii. $\frac{1}{8}$ iii. $\frac{2}{8}$ iv. $\frac{4}{8}$

4. a. $\frac{4}{9}$ b. $\frac{1}{3}$ c. $\frac{5}{9}$ d. $\frac{2}{3}$
5. a. $\frac{240}{702}$ b. $\frac{132}{702}$ c. $\frac{272}{702}$ d. $\frac{487}{702}$ e. $\frac{312}{702}$
6. a. $\frac{1}{3}$ b. $\frac{2450}{3540}$ c. $\frac{1770}{3540}$ d. $\frac{600}{3540}$ e. $\frac{870}{3540}$
7. a. $\frac{1}{4}$ b. $\frac{1}{2}$ c. $\frac{1}{6}$ d. $\frac{1}{2}$ e. $\frac{1}{2}$

Exercise 39d

1. a. 53 b. 13 c. 160 d. 121 e. $\frac{121}{160}$ f. $\frac{60}{160}$
   g. $\frac{40}{160}$ h. $\frac{39}{160}$ i. $\frac{21}{160}$ j. $\frac{128}{160}$ k. No-one is in both groups
   l. There are nineteen people who do both m. 'Being unemployed'
   n. 'Belong to Party Z' o. 'Belonging to Party Y or Party Z' p. $\frac{54}{160}$
2. b, c
3. a. 0.975 b. 0.0287 c. 0.9713 d. 0.00082369
4. a. {3, 4, 5, 6, 7, 8} b. {1, 2, 6, 7, 8} c. {3, 5, 6, 7} d. {1, 2, 4, 5, 7, 8} e. ø
5. a. (E_1, E_3), (E_2, E_3), (E_2, E_4), (E_3, E_5) b. (E_2, E_4), (E_3, E_5)
   c. i. $\frac{1}{4}$ ii. $\frac{1}{4}$ iii. $\frac{3}{8}$ iv. $\frac{3}{4}$ v. $\frac{5}{8}$ vi. $\frac{3}{8}$
   vii. $\frac{7}{8}$ viii. 0 ix. $\frac{3}{8}$ x. $\frac{3}{4}$ xi. $\frac{5}{8}$ xii. $\frac{7}{8}$
   xiii. 1 xiv. 0 xv. $\frac{3}{8}$
6. a. $\frac{5}{8}$ b. $\frac{5}{8}$ c. $\frac{3}{4}$ d. $\frac{3}{4}$ e. $\frac{6}{8}$ f. $\frac{6}{8}$
   g. $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
7. a. i. $\frac{2}{3}$ ii. $\frac{7}{12}$ iii. $\frac{3}{4}$ iv. $\frac{8}{15}$ v. 0
   vi. $\frac{4}{5}$ vii. 0 viii. $\frac{4}{5}$ ix. $\frac{1}{3}$ x. $\frac{1}{5}$
   b. 60 c. $P(A) + P(B) \neq 1$

Exercise 39e

1. a, b 2. none are independent 3. There are many possible answers for each
4. a. i, ii are independent b. i. $\frac{4}{90}$ ii. $\frac{16}{90}$
5. The probability of A and B occurring is not equal to the product of the probability of A and the probability of B.
6. a. independent: (E_1, E_2), (E_1, E_4), (E_2, E_3), (E_3, E_4)
   complementary: (E_1, E_3), (E_2, E_4)
   mutually exclusive: (E_1, E_3), (E_2, E_4), (E_3, E_5), (E_2, E_5)
   b. i. First marble is red ii. Second marble is red
   iii First marble is black and second marble is red
   iv First and second marbles are black
   v. First marble is red or second marble is black
   c. i. $\frac{2}{5}$ ii. $\frac{2}{5}$ iii. $\frac{3}{5}$ iv. $\frac{3}{5}$ v. $\frac{6}{25}$ vi. $\frac{16}{25}$
   vii. $\frac{4}{25}$ viii. $\frac{19}{25}$ ix. $\frac{6}{25}$ x. 0 xi. 1 xii. $\frac{6}{25}$
   xiii. $\frac{21}{25}$ xiv. $\frac{2}{5}$ xv. $\frac{3}{5}$ xvi. $\frac{6}{25}$ xvii. $\frac{9}{25}$ xviii. $\frac{19}{25}$
   xix. $\frac{4}{25}$ xx. $\frac{2}{5}$

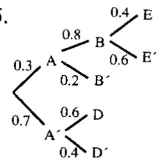
Chapter 39 – Problems and Investigations

1. $\frac{6}{14}$ 2. $\frac{1}{7}$ 3. $\frac{4}{7}$

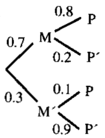
Chapter 40: Conditional Probability

Exercise 40a

1. 0.3 2. 0.95 3. 0.6 4. 0.035 5.
6. 0.4 7. 0.4 8. 0.42 9. 0.144



10. a.



b. 0.59

Exercise 40b

1. 0.1 2. 0.4 3. 0.9 4. 0.81 5. 0.06

6.

	Apple	Acorn	PC	Other	Total
Junior	25	10	32	33	100
Senior	40	8	30	22	100
Total	65	18	62	55	200

7. $\frac{65}{200}$ 8. $\frac{30}{100}$
9. $\frac{10}{100}$ 10. $\frac{33}{55}$

Exercise 40c

1. 0.8 2. 0.9 3. 0.02 4. 0.34 $\frac{16}{34}$ 6. $\frac{2}{66}$
7. $\frac{5}{100}$ 8. a. 0.2325 b. 0.6458 9. a. 0.51 b. $\frac{51}{88}$
10. a. 0.45 b. $\frac{32}{45}$ c. $\frac{32}{69}$

Chapter 40 – Problems and Investigations

1. a. 0.296 b. $\frac{186}{296}$ c. $\frac{246}{540}$ 2. a. $\frac{187}{245}$ b. $\frac{60}{137}$

Chapter 41: Random Numbers and Simulations

Exercise 41a

1. a. $\frac{1}{27}$ b. $\frac{6}{27}$ c. $\frac{19}{27}$ d. $\frac{6}{27}$ e. $\frac{17}{27}$
   f. $\frac{8}{27}$ g. $\frac{8}{27}$ h. 0 i. $\frac{26}{27}$ j. $\frac{9}{27}$
2. a. $\frac{1}{10}$ b. $\frac{16}{100}$ c. $\frac{7}{100}$ d. $\frac{16}{100}$ e. $\frac{45}{100}$
3. a. $\frac{1}{100}$ b. $\frac{4}{100}$ c. $\frac{1}{10}$ d. $\frac{1}{1000}$ e. $\frac{720}{1000}$

Exercise 41b

1. about 16% 2. about 7% 3. more than 90%

Exercise 41c

1. about 30% 2. about 50%

Chapter 42: The Normal Distribution

Exercise 42a

1. 0.1915 2. 0.3888 3. 0.4953 4. 0.1359 5. 0.4194 6. 0.1286
7. 0.3049 8. 0.1151 9. 0.1284 10. 0.2978 11. 0.3686 12. 0.1630
13. 0.0161 14. 0.5028 15. 0.9469 16. 0.1255 17. 0.4725 18. 0.2260
19. 0.6217 20. 0.6179 21. 0.6368 22. 0.9032 23. 0.0000 24. 0.9332

Exercise 42b

1. 0.42 2. 1.16 3. 1.652 4. 1.234 5. 0.70 6. 0.661
7. 1.587 8. -1.355 9. 0.524 10. 1.036 11. 0.516 12. 0.641
13. 1.099 14. -0.129 15. 0.019

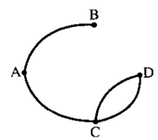
Exercise 42c

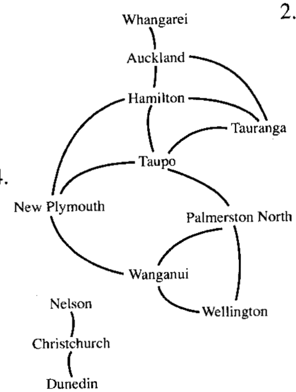
1. a. 0.5 b. 0.2734 c. 0.0987 d. 0.1747 e. 0.4013 f. 0.5987
   g. 0.1974 h. 0.2413 i. 0.3721 j. 0.1056
2. a. 0.3094 b. 0.4452 c. 0.2569 d. 0.1292 e. 0.9834
3. a. 5 b. 6.572 c. 8.108 d. 6.155 e. 7.022 f. 8.24
   g. 7.316 h. 2.018 i. 4.625 j. 3.251
4. a. 12.88 b. 11.935 c. b = 13.042, a = 2.74
   d. 3.157 e. a = 3.157, b = 10.843
5. a. 0.0913 b. 0.9849 c. 0.3362 d. 39.25 kg e. 33.238 kg
6. a. 0.5403 b. 0.9690 c. 16.375
7. a. 0.0548 b. 0.4516 c. 23 rides
8. a. 0.9911 b. 99.11% c. 60.38% d. 27.94 minutes

Chapter 43: Networks

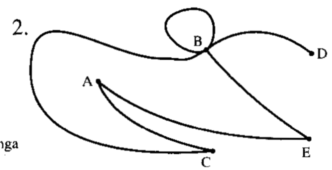
Exercise 43a

1.

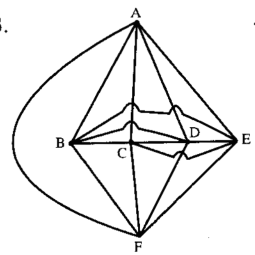




2.

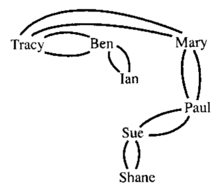


3.



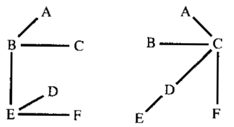
4.

5.

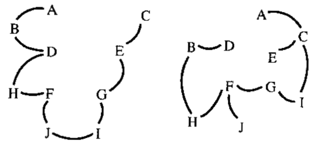


Exercise 43b

1.

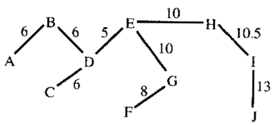


2.



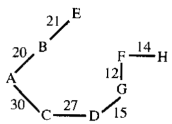
Exercise 43c

1. a.



b. 74.5

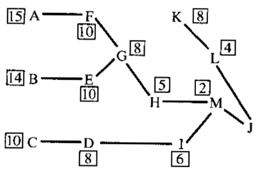
2. a.



b. 139 m

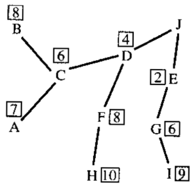
Exercise 43d

1.



Route is A – F – G – H – M – J
Total distance 15

2.



Route is A – C – D – J
Total distance 7

Exercise 43e

1. Every vertex has an even number of edges.
   F – A – B – G – F – B – C – D – E – H – C – E – F

2. There are exactly 2 vertices with an odd number of edges.
   A – F – E – H – D – B – G – A – B – C – D – E

3. a. $A \overset{5}{-} G \overset{5}{-} D \boxed{\overset{3}{-} C \overset{3}{-}} D \overset{5}{-} E \overset{5}{-} F \overset{3}{-} A \overset{4}{-} B \overset{5}{-} C \overset{4}{-} G \overset{4}{-} E$
   Double at DC, which is the least possible doubled distance.
   b. 46

4. a. $F \overset{5}{-} G \overset{5}{-} C \overset{4}{-} D \boxed{\overset{3}{-} H \overset{3}{-}} D \overset{7}{-} E \overset{6}{-} H \overset{4}{-} C \overset{4}{-} B \overset{4}{-} G \overset{3}{-} E \overset{3}{-} F \overset{5}{-} A \overset{4}{-} B$
   Double at DH, which is the least possible doubled distance.
   b. 60

5. a. $C \overset{5}{-} A \overset{6}{-} B \overset{2}{-} K \overset{2}{-} B \overset{4}{-} D \overset{3}{-} I \overset{2}{-} H \overset{2}{-} I \overset{4}{-} J \overset{4}{-} G \overset{2}{-} F \overset{2}{-} G \overset{3}{-} H \overset{3}{-} E \overset{4}{-} F \overset{6}{-} C \overset{3}{-} K \overset{4}{-} E \overset{2}{-} D$
   b. 63

Exercise 43f

1. a.

Task	Time to complete	Depends on
A	3	–
B	2	–
C	4	A
D	2	A
E	2	B, C
F	3	B, C
G	4	B, C
H	3	D, E
I	2	F
J	1	G
K	4	H
L	1	I, J, K

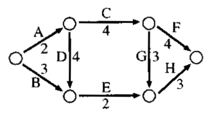
b.

Task	Time to complete	Depends on
A	2	–
B	3	–
C	3	–
D	4	A
E	3	B
F	3	C
G	2	D, E
H	2	G, F

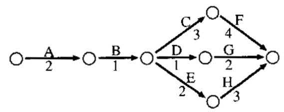
c.

Task	Time to complete	Depends on
A	2	–
B	3	A
C	2	A
D	4	B
E	2	B
F	1	C
G	2	D
H	3	E
I	1	F

2. a.

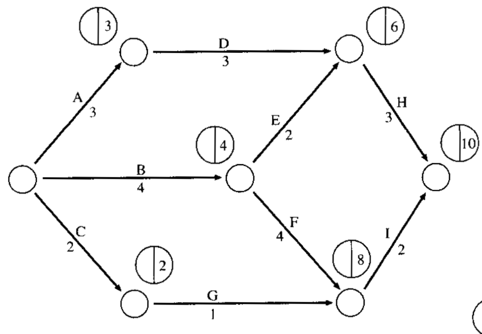


b.



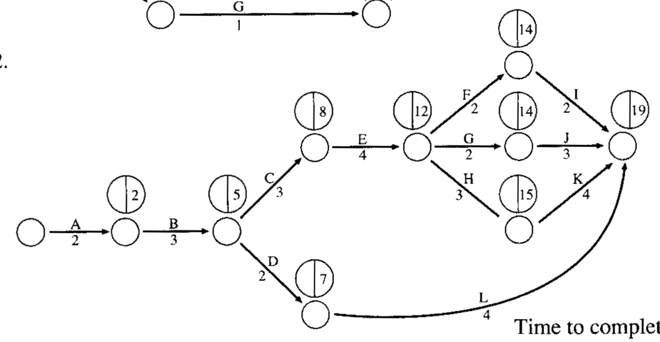
Exercise 43g

1.



Time to completion is 10
Critical Path is B – F – I

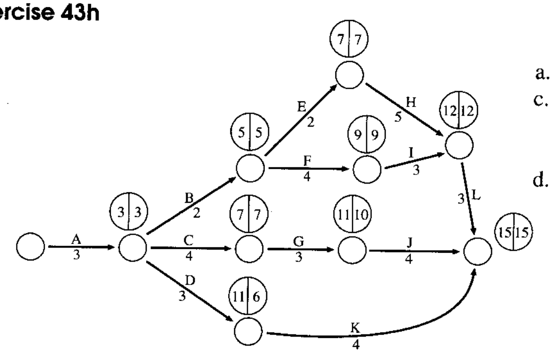
2.



Time to completion is 19
Critical Path is A – B – C – E – H – K

Exercise 43h

1\.



a. Time to completion is 15
c. Critical Path is either
A – B – E – H – L or
A – B – F – I – L
d. 5

2\. m = 4, n = 5, p = 11, q = 3, r = 5, x = 1

Chapter 44: Polar Co-ordinates

Exercise 44a

3\. A (15, 22°), B (13, 77°), C (10, 142°), D (14, -122°), E (14, -19°)
4\. a. (4.5, 117°) b. (5.8, 59°) c. (5, -37°) d. (6.4, -129°) e. (5.4, -22°)

Exercise 44b

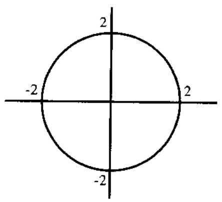
1\. a. (4.1, 2.9) b. (5.1, 4.8) c. (-2.1, 4.5) d. (6.1, -5.1) e. (-7.6, -4.8)
2\. a. (6.4, 51.3°) b. (5, 126.9) c. (6.7, -153.4) d. (5.4, -111.8) e. (6.4, -38.7°)

Exercise 44c

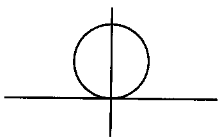
1\. a. (5.4, -21.8°) b. (5.8, -59°) c. (6.4, 128.7°) d. (5.6, 26.6°) e. (2.4, 123.0°)
2\. a. (5.7, 4.1) b. (6.0, 5.6) c. (-2.5, 4.8) d. (-6.1, -13.7) e. (1.8, -2.0)

Chapter 44 – Problems and Investigations

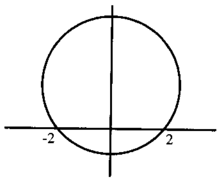
1\. a. (20, 30°)
b. i. (25, 47°) ii. (15, 155°) iii. (5, -52°) iv. (19, -125°)
2\. a. (20, 58°) b. (35, -29°) c. (5, -122°) d. (17, 163°)
3\. a. (226, 118°) b. (258, 14°) c. (327, -15°)
4\. a.



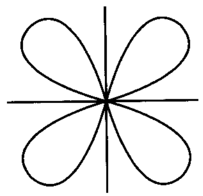
b.



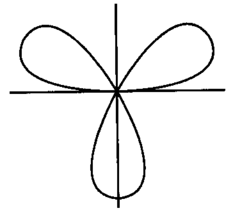
c.



d.



e.



GLOSSARY

Absolute Value (p. 111): Another expression for modulus.
Acceleration (symbol a) **(p. 241):** The instantaneous rate of change of speed with respect to time.
Acute Angle (p. 202, 211): An angle of size less than 90°.
Adding (symbol +) **(p. 1):** An arithmetic operation. Adding rational expressions, **p. 31.**
Adjacent (p. 364): Nodes joined by an edge are said to be adjacent.
Advertiser's Scale (p. 309): A slanting scale used to create favourable impressions.
Algebraic Conventions (p. 1, 5): See BEDMAS.
Algebraic Expressions (p. 4): A collection of algebraic terms.
Algebraic Fractions (p. 31): A rational expression with variables.
Algebraic Method (p. 123): Solving a problem using algebra. For straight lines, **p. 11, 60–62**; for straight lines and circles, **p. 123–124.**
Algebraic Term (p. 4): A product of numbers (called *coefficients*) and letters (called *variables*). Each of the variables may be raised to a *power*.
All Science Teachers Cry (p. 212): A mnemonic used to remember the quadrants in which the different trigonometric functions have positive values.
Amplitude (p. 218): The magnitude of the coefficient of a sine or cosine function.
Annulus (p. 40): A disc with a smaller disc removed, where both discs have the same centre.
Antiderivative (p. 235, 245): The result of antidifferentiating.
Antidifferentiation (p. 235): also called integration. The opposite process to differentiation. Rules for antidifferentiation, **p. 235**; notation with, **p. 236**; of expressions, **p. 235–236.**
Applications: Of straight line equations, **p. 92.**
Arc (p. 230): Part of the circumference of a circle.
Area (p. 208): Under a curve, **p. 246–250**; of a trapezium, **p. 250.**
Area Rule (p. 208): A relationship used to find the area of any triangle given the length of two sides and the angle between them.
Arithmetic Progression (p. 154): See arithmetic sequence.
Arithmetic Sequence (symbol AP) **(p. 154):** A sequence with a constant difference between adjacent terms.
Associative Law (p. 5): For + or ×, the use of brackets does not affect the result, eg $(3 + 4) + 5 = 3 + (4 + 5)$
Asymptote (p. 54): Line which a graph gets closer and closer to, but never touches.
Average (p. 282): The mean of a sample.
Average Speed (p. 187): Also called average velocity. The average rate of change of distance, with respect to time.

Bar Graphs (p. 262): Statistical graphs where vertical bars represent frequencies.

Base Changing Rule (p. 144): Used to change from one base to another.
BEDMAS (p. 5): A mnemonic which gives the correct order in which arithmetic operations are done.
Bias (p. 266): When the conclusions from a survey do not describe the characteristics of the whole population.
Box Plots (p. 302): A form of data display.

Calculators, use of with: Repeated substitution, **p. 3**; linear equations, **p. 14, 17**; formulae, **p. 44**; graphs, **p. 47, 49, 52, 106**; simultaneous equations, **p. 63, 100**; quadratic equations, **p. 73**; circles, **p. 121**; indices, **p. 132**; sequences, **p. 151**; gradients, **p. 166**; trigonometric equations, **p. 177**; derivatives, **p. 178**; graphs of the trigonometric functions, **p. 180**; tangents, **p. 181**; turning points, **p. 185**; trigonometry, **p. 198**; definite integrals, **p. 246**; area under curves, **p. 248**; standard deviations, **p. 249**; measures of spread, **p. 287**; simulation of random events, **p. 349–350**; polar co-ordinates, **p. 384–385.**
Calculus (p. 174): Differentiation, integration and associated processes and graphs.
Census (p. 256): A survey where all the target population is surveyed.
Central Tendency (p. 281): The central value of a sample or population.
Circle: Equation of, **p. 120–122**; intersecting with a line, **p. 123.**
Circular Functions (p. 212): Another name for the trigonometric functions.
Clusters (p. 268): Randomly chosen representative subsets for collecting a sample from.
Coefficient (p. 4): A number which multiplies a variable in a term.
Collinear: 3 or more points which lie on the same straight line.
Common Denominator (p. 33): A denominator common to two or more fractions.
Common Difference (p. 154): The constant difference associated with an arithmetic sequence.
Common Factors (p. 26): Factors which divide into each term of an expression without remainder.
Common Ratio (p. 158): The constant value associated with a geometric sequence.
Commutative Law (p. 5): The order of two elements being combined by addition or subtraction does not affect the result.
Complementary Events (p. 333): Events which are mutually exclusive and where each outcome of a trial belongs in one of the events.
Compresses (p. 103, 218): Makes smaller.
Compression (p. 277): Shortening of the horizontal axis.
Concurrent (p. 100): Lines which go through the same point are concurrent.
Conditional Probability (p. 339): Probability over a subset of a sample space.
Congruent (p. 107): Identical in shape and size.
Connected Network (p. 364): Network with all nodes accessible along edges.
Constant (p. 70): A number (which does not vary).

Constant of Integration (p. 235): A constant which results from integrating. Finding the constant, **p. 239.**
Continuous Data (p. 277): Data for which another possible value lies between any two values. On a cumulative frequency graph, **p. 296.**
Continuous Function (p. 51): A function whose graph is an unbroken line.
Continuous Population (p. 354): A population of values which are continuous. See continuous data.
Cosine (symbol cos) **(p. 197):** A trigonometric function.
Cosine Rule (p. 206): A relationship between three sides and one angle of a triangle.
Cover Up Rule (p. 115): Used for finding the horizontal asymptote of a hyperbola.
Cube Root (p. 131): A number which when cubed gives the original number.
Cubic (p. 68): A polynomial of degree 3.
Cubic Functions (p. 109): An expression of the form $y =$ *cubic expression.*
Cubics: Drawing graphs of **p. 50–51**; sketching graphs of, **p. 109–110.**
Cumulative Frequency Graph p. 295: A graph showing the number of a sample or population with a value less than, or less than or equal to, a particular value. Also called *ogives*.
Cycles (p. 365): Paths in a network that begin and end at a common vertex.

Data (p. 253): The numbers and information associated with statistics.
Data Display: Discrete, ungrouped data, **p. 273**; discrete, grouped data, **p. 274**; continuous data, **p. 277**; data with non-equal intervals, **p. 279.**
Decimal Places (p. 2): The number of digits after the decimal point in a decimal number.
Decreasing Function (p. 50, 182): A function whose 'y' values decrease as its 'x' values increase.
Deduction (p. 272): A conclusion drawn from a survey.
Definite Integral (p. 245): An integral calculated between two limits.
Degree (p. 68, 70, 109): The highest power of the variable in any of the terms of a polynomial.
Denominator (p. 29): The *integer* or *variable* below the horizontal line in a fraction.
Derivative (p. 174, 188): Gradient function.
Difference (p. 27): The result of subtracting two terms or numbers. Of two squares, **p. 27**; of two cubes, **p. 28.**
Differentiating: simple, **p. 174**; with brackets, **p. 175**; negative indices, **p. 177**; rational powers, **p. 177.**
Differentiation (p. 174): Finding a derivative.
Differentiable (p. 174): A function which can be differentiated.
Direct Proportion: Two quantities x and y are in direct proportion if their *quotient* $\frac{x}{y}$ is constant.
Disconnected Network (p. 364): Network in which some pairs of nodes are not accessible along edges.

Discontinuous Function (p. 51): A function whose graph has a break.
Discontinuity (p. 51): A point where a graph is no longer continuous.
Discrete Data (p. 272): Data where no other values are possible between the values obtained in the data. On a cumulative frequency graph, **p. 296–297.**
Discriminant (p. 75): A number calculated from the constants of a quadratic.
Dispersion (p. 281): Also called spread.
Distance between two points (p. 79)
Distributive Law (p. 5, 24): A law describing how brackets are used.
Dividing (symbol ÷) **(p. 1):** An arithmetic operation. Dividing rational expressions, **p. 32.**
Divisor (p. 32): The term 'doing' the division.
Domain (p. 46, 114): The set of first elements in the ordered pairs of a relation. Of a trigonometric function, **p. 218.**

Edges (p. 364): Lines between nodes in a network.
EFT (p. 376): Earliest finish time for a task.
Elimination (p. 60): A method of solving simultaneous equations.
Empty Set (p. 333): A set with no elements in it.
Enlarge (p. 218): Make larger.
Enumeration (p. 150): One way of writing a sequence, by listing.
Equation (p. 11): A *mathematical sentence* involving an equals sign and at least one variable. Trigonometric, **p. 214–216**; involving radians, **p. 225**; linear, **p. 11**; simultaneous, **p. 60–63**; of a straight line, **p. 87–91**; more complicated straight line equations, **p. 82**; quadratic, **p. 102–106.**
Equilateral Triangle (p. 203): A triangle with all three sides the same length.
Equiprobable (p. 325): Events with equally likely outcomes.
Erratic or residual variation (p. 315): Variations in time series that are not secular or periodic.
Estimate (p. 292): An approximation to an exact value.
Event (p. 323): A possible outcome of a statistical trial.
Expanding (p. 24, 236): Removing brackets from an expression by multiplying. When antidifferentiating, **p. 236–237.**
Exponent (p. 1, 4): Using exponents, **p. 7 – 8.**
Exponential Function (p. 138, 142): Inverse of the natural logarithmic function.
Extrapolating beyond the Data (p. 308): A trend on a graph is continued into the future.

Factorising (p. 26): Rewriting a mathematical expression as a product of simpler factors. Factorising quadratics, **p. 26.**
Float (p. 377): The difference between the LST and the EST for a task.
Formula (Formulae, plural) **(p. 1, 37):** An algebraic equation in which the mathematical relationship between different variables is expressed. Changing the subject of, **p. 41**; using formulae, **p. 43–44**; and sequences **p. 150.**

Fraction: A number expressed as a quotient. The top number is called the *numerator*, the bottom the *denominator*. An *algebraic fraction* or rational expression is a fraction with variables **(p. 31)**.
Free of bias (p. 266): When each member of the population is equally likely to be included in the sample.
Frequency (p. 272): The number of times a value occurs in a survey.
Frequency Histogram (p. 277): A diagram used to illustrate a continuous population.
Frequency Table (p. 272): A table in which the frequency of events or data is recorded.
Function (p. 46): A relation in which each element of the domain appears in only one of the ordered pairs of the relation.

General Term (p. 148): Also called the nth term.
Geometric Progression (p. 158): A geometric sequence.
Geometric Sequence (symbol GP) **(p. 158):** A sequence where each term gives a constant value when divided by the previous term.
Gradient (p. 81, 183, 164–166, 167): The ratio of the $\frac{\text{vertical change}}{\text{horizontal change}}$ of a straight line. Relationship with angles, **p. 85**; of a tangent, **p. 179, 183**; of a normal, **p. 179**; on a distance-time graph, **p. 187–188.**
Gradient Function (p. 169): A function which maps a point onto its gradient.
Graph (p. 46): A set of ordered pairs plotted on a grid. Of a sequence, **p. 151.**
Graphical method: Solving a problem using a graph. For straight lines, **p. 55, 92, 99**; for straight lines and circles, **p. 123.**
Graphs: Drawing graphs, **p. 48–51**; sketching graphs, **p. 48–52**; of parabolas, **p. 102–106**; of cubics, **p. 109–110**; of the modulus function, **p. 98–99**; involving square roots **p. 113**; of hyperbolas, **p. 114–116**; of trigonometrical functions, **p. 217–220.**
Grouped Data (p. 274): Data displayed in groups.

Histograms (p. 277): Bar graphs on which continuous data are displayed.
Horizontal Asymptote (p. 114): A horizontal line which a graph approaches but does not cross. The value which is excluded from the range of a hyperbola.
Hyperbola (p. 114): The graph of a function of the type $y = \frac{Ax + B}{Cx + D}$.
Hypotenuse (p. 38, 79): The side of a right-angled triangle opposite the right angle.

Image (p. 142): The object after it has been transformed.
Increasing Function (p. 50, 182): A function whose 'y' values increase as its 'x' values increase.
Independent (p. 347): Uninfluenced by preceding outcomes.
Independent Events (p. 336): Two events where the outcome of one has no effect on the outcome of the other.

Index (p. 1, 4): An alternative expression for exponent. Working with indices, **p. 126–131.**
Inequation (p. 21): A mathematical sentence involving an inequality.
Initial Line (p. 380): The positive horizontal direction from which angles are measured in polar co-ordinates.
Instantaneous Velocity (p. 188): The velocity at a particular instant.
Integral (p. 235): Another word for antiderivative. And area under a curve, **p. 246–247.**
Integration (p. 235): See antidifferentiation.
Intercept (p. 48, 103): The point where two or more lines meet.
Interquartile Range (p. 285): The difference between the upper and lower quartiles.
Intersection (p. 333): The set containing the elements common to two or more other sets.
Interval Notation (p. 171): a notation for subsets of the real numbers, using brackets.
Interviewers (p. 258): The people who conduct a survey.
Inverse Function (p. 138, 142): The function which reverses the effect of another function.
Inverse Relation (p. 46): The relation obtained by reversing elements within each ordered pair of a relation.
IQ Test (p. 355): A test to determine human intelligence whose outcome is a number.
Irrational Number (p. 222): A number which cannot be written as the quotient of two integers.
Isosceles Triangle (p. 39, 80): A triangle with two sides of equal length.

Like Terms (p. 4): Terms which may be combined by adding or subtracting.
Limit: The value a function *approaches*. And sequences, **p. 152.**
Line (p. 87): Applications of straight line equations, **p. 92**; intersection with another line, **p. 99**; intersection with a circle, **p. 123.**
Line Segment (p. 97): A line between two points.
Linear (p. 11): An equation or polynomial in which the highest power of any variable is 1.
Listing (p. 150): Also called enumeration.
Local Maximum (p. 182, 185): A turning point where a function changes from increasing to decreasing.
Local Minimum (p. 183, 185): A turning point where a function changes from decreasing to increasing.
Loops (p. 364): Edges that connect a node to itself.
Lower Quartile (p. 283): The value for which 25% of the sample or population is less than or equal to. Also called the 25th percentile. On a box plot, **p. 302.**
LST (p. 377): Latest start time for a task.

m (p. 81): Commonly used symbol for gradient.
Mapping Notation (p. 49): A way of representing functions.
Market Research (p. 257): A poll taken for commercial purposes.
Mathematical Modelling (p. 18, 37): The process where problems described in words are changed into equations.
Matrix (p. 26): An array of numbers in rows and columns (plural matrices).
Maxima (p. 103, 182–185): The maximum value of a quantity or function.
Maximum Turning Point (p. 50, 103, 182–185): A turning point where a function changes from increasing to decreasing.
Mean (p. 282): The sum of all the values divided by the total number of values.
Median (p. 283): The middle value of a population or sample. On a cumulative frequency graph, **p. 296**; on a box plot, **p. 302**.
Midpoint (p. 97): The point halfway between the end-points of a line segment.
Minima (p. 103, 182–185): The minimum value of a quantity or function.
Minimum Spanning Tree (p. 367): A spanning tree with lowest edge sum.
Minimum Turning Point (p. 50, p. 103, p., 182–185): A turning point where a function changes from decreasing to increasing.
Mode (p. 284): The most frequently occurring value.
Modulus (symbol $|x|$) **(p. 111):** The function which makes all numbers positive.
Moving Averages (p. 316): Method of analysing secular trends in time series data.
Multiplying (symbol ×) **(p. 1):** An arithmetic operation. Multiplying rational expressions, **p. 31**.
Mutually Exclusive (p. 332): Two events which cannot happen together.

Natural Logarithm (p. 136): A function with domain $\{x: x > 0\}$, ie defined for all positive numbers. General properties, **p. 137–138**.
Nature (p. 75): The number and type of roots of a quadratic equation.
Nature of a Turning Point (p. 50): Whether a turning point is a maximum or a minimum.
Network (p. 364): Diagram consisting of nodes and edges, used to represent information. Networks containing numbers, **p. 366–368**.
Nodes (p. 364): Points in a network.
Normal (p. 179): A line perpendicular to a tangent.
Normal Distribution (p. 354–363): The distribution which describes heights, lengths, weights and many other features of nature.
Normal Table (p. 355 and appendix): A table giving the areas under normal probability curves. See appendix.
nth term (p. 148): The member of a sequence associated with n.
Numerator (p. 29): The integer or variable above the horizontal line in a fraction.

Ogives (p. 295): See cumulative frequency graph.

Parabolas (p. 70, 102–106): Graphs of quadratic functions.

Parallel lines (p. 83, 86, 95): Lines with the same gradient.
Percentile (p. 284): Generalisations of the median and quartiles. 25th Percentile, also called the lower quartile, **p. 283**; 50th percentile, also called the median, **p. 284**; 75th percentile, also called the upper quartile, **p. 284**; on a cumulative frequency graph, **p. 296–297**.
Period: Of a trigonometric function, **p. 218**.
Periodic Movements (p. 315): Variations about a general trend in a time series.
Perpendicular Lines (p. 95–96): Lines at right angles to each other.
Pie Graphs (p. 300): Graphs used to show proportions.
Piloting (p. 257): Pre-testing the questions to be used in a survey.
Plotting (p. 48): Transferring the ordered pairs of a relation accurately onto a graph.
Point of Inflection (p. 54, 170, 185): A point where the derivative has a maximum or minimum value.
Point Symmetry (p. 50): A graph which can be rotated through 180° about a point onto itself.
Polar Co-ordinates (p. 380): method of describing a point's position using distance and angle.
Pole (p. 380): Reference point for polar co-ordinates.
Political Poll (p. 257): A survey used to obtain political information.
Poll (p. 257): A survey which seeks opinions from people.
Polynomial (p. 68–69, 70): A function which is the sum of several terms.
Population (p. 281): The values of interest in a survey or experiment.
Population Mean (symbol μ) **(p. 355, 282)**
Power (p. 1, 4, 8): An alternative expression for exponent.
Power Form (p. 7–8): Any number written as a power of another number.
Primes: Numbers that have only two factors.
Probability (p. 323): A measure of how certain it is that an event will occur. Calculating probabilities for normal distributions, **p. 360–361**.
Probability Density Graph (p. 354): The limiting graph as the sample size gets larger of a relative frequency histogram.
Probability Tree (p. 329): A diagram used to calculate probabilities.
Proof (p. 131, 198, 202, 206): Statements showing that a relationship is correct.
Proportion (p. 276): A comparison of different amounts.
Pythagoras' Theorem (p. 79, 120, 198–199): The square of the hypotenuse equals the sum of the squares of the other sides of a right-angled triangle.

Quadrant (p. 211): One of four regions defined by the x and y axes.
Quadratic (p. 26, 68): A polynomial of degree 2. Drawing graphs of **p. 47–48**; sketching graphs of, **p. 102–106**; on a cumulative frequency graph, **p. 259, 260**.
Quadratic Equation (p. 70): Solution of, **p. 71–72**; roots of, **p. 74–75**.
Quadratic Expression (p. 25): A mathematical expression which can be written in the form $Ax^2 + Bx + C$, where x is a variable and A, B and C are constants.

Quadratic Function (p. 70, 102): A function of the form $y = \textit{quadratic expression}$.
Quartiles (p. 283): Two numbers associated with the central tendency of samples and populations.
Questionnaire (p. 259): A series of written questions.
Quotient (p. 29): An expression with a denominator and numerator; the result of dividing two numbers or variables.

Radians (p. 222): A natural unit for measuring angle size.
Random Numbers (p. 346): A set of numbers not containing any predetermined pattern.
Random Selection (p. 266, 268): A method for collecting a sample in which every member of the population has the same chance of being selected.
Range (p. 46, 114): The set of second elements from a relation. Of a trigonometric function, **p. 218.**
Range (p. 285): A simple measure of spread.
Rate of Change (p. 92, 164–165): The gradient of a line or curve. Of distance, **p. 187**; other quantities, **p. 188.**
Ratio (p. 158): A comparison of two or more quantities.
Ratio to Moving Average (p. 318–319): Technique used to analyse periodic variation in a time series.
Rational Expression (p. 31): Any quotient consisting of two algebraic expressions.
Rational Powers (p. 131): A power where the index is a rational number.
Raw Data (p. 272): Data at an early stage of analysis.
Real Numbers (p. 49): The largest set of numbers that can be represented on a number line.
Rearranging (p. 41, 70): Changing the order of terms in an expression, often so that another variable becomes the subject.
Reciprocal (p. 6, 32): A multiplicative inverse. The reciprocal of $\frac{x}{y}$ is $\frac{y}{x}$.
Reciprocal Powers (p. 133): Used for solving equations with indices.
Recursively (p. 150): One way of writing a sequence.
Relation (p. 46): A set of ordered pairs, usually numbers.
Relative Frequency (p. 276): The proportion of total occurrences of any value.
Relative Frequency Histogram (p. 354): Histogram showing relative frequency of outcomes.
Respondent (p. 259): The person being asked questions in a survey.

Sample (p. 253, 266, 281): A small part of a population being investigated.
Sample Mean (symbol $\bar{x}$) **(p. 282):** The mean value of a sample.
Sample Space (p. 324): A set whose elements describe every possible outcome of a trial.
Sample Standard Deviation (symbol s) **(p. 286):** The standard deviation of a sample.

Sample Survey (p. 256): A survey where a small section of the target population is investigated.
Sampling (p. 256): Taking a subset of a population.
Scatter Diagram (p. 306–307): Graph used with paired data.
Sector (p. 230): A region formed by two radii and an arc.
Secular Trend (p. 316): A general tendency for increase, decrease or stability in time series.
Sequence (p. 148): A function which associates a real number with each natural number.
Series (p. 153): The sum of the successive terms of another sequence.
Set Builder Notation (p. 47): Symbols used to describe sets and relations.
Sigma Notation: (symbol Σ) **(p. 153):** Notation associated with sequences and series.
Simplifying (p. 4): Changing an algebraic expression into a simpler form.
Simultaneous Equations (p. 60, 99, 123): Equations, each of which contains the same variables, and which usually have a common solution.
Sine (symbol sin) **(p. 197):** A trigonometric function.
Sine Rule (p. 208): The relationship between one side of a triangle and its opposite angle, and a second side and its opposite angle.
Sketch Graph (p. 102): A quickly drawn graph of a function showing the key features.
Slope (p. 81, 164, 165): Another expression for gradient.
Solution Sets (p. 22): The set of solutions to an equation or inequation,
Solving (p. 11): Finding the values of the variables which make an equation or inequation true. Solving equations, **p. 11**; solving inequations **p. 21.**
Spanning Tree (p. 366): A tree including all vertices and whose edges are in the network.
Speed (symbol v) **(p. 187, 241):** Also called velocity. The rate of change of distance with respect to time.
Spread (p. 281): How far the data is spread from the central value.
Spreadsheet (p. 15, 44, 191, 193): Numerical solution method for solving equations etc by computer. Use in: statistics, **p. 287-288**; data display, **p. 305–306**; time series, **p. 319-321**; simulating random events, **p. 348–349, 351.**
Square Root (p. 131): A number which when squared gives the original number.
Standard Deviation (symbol s or σ) **(p. 286):** A measure of spread. The square root of the variance.
Statistics (p. 253): The collection, display and analysis of data.
Stem and Leaf Diagrams (p. 303–304): Diagram used in data display, similar to tally charts.
Strata (p. 267): Important groups in a population which must be represented in a survey.
Stratified Sampling (p. 267): Choosing a sample so that subgroups of the population are proportionally represented.

Structured Questions (p. 259–261): Questions where the answer falls into predetermined categories.
Subject (p. 41): The variable appearing on its own on one side of the equals sign in an equation or formula.
Substitution (p. 1, 43, 62, 66): The process of replacing variables with numbers or other variables.
Substitution Method (p. 62): A method of solving simultaneous equations.
Subtend (p. 199, 230): Lie under.
Subtracting (symbol –) **(p. 1):** An arithmetic operation. Subtracting rational expressions, **p. 33**.
Sum (p. 27): The result of adding two or more numbers or like terms. Of two cubes, **p. 28**; of the first n terms of an arithmetic progression, **p. 157–157**; of a geometric progression, **p. 160**; sum to infinity, **p. 161–162**; of rational expressions, **p. 33**.
Suppressed Scale (p. 309): A vertical scale with points plotted over a smaller range of values.
Surd (p. 202): The root of a rational number which is not rational itself, such as $\sqrt{2}$.
Survey (p. 255): A statistical investigation in which data is collected. Purpose of, **p. 257**; execution of, **p. 258**.
Systematic Method (p. 267): A quick method for collecting a sample.

Table (p. 47): Used to represent some of the ordered pairs of a relation.
Tallies (p. 272): Vertical and diagonal lines used to record frequencies.
Tally Chart (p. 272): A table in which tallies are displayed.
Tangent (p. 179, 123): A straight line that touches a curved line in one place. To graph, **p. 169, 179–181**.
Tangent (symbol tan) **(p. 197):** A trigonometric function.
Target Population (p. 255): The population being investigated in a survey.
Term (p. 37, 68, 148): One member of a sequence or expression.
Time Series (p. 314): A sequence of values of a variable over time.
Time to Completion (p. 376): Longest path in a critical path diagram.
Trapezium (p. 250): A quadrilateral with one pair of parallel opposite sides.
Tree (p. 366): A connected network without cycles.
Trial (p. 324): One repetition of an experiment.
Triangles: labelling of, **p. 206**.
Triangular Brackets (p. 150): Used to write a sequence.
Trigonometric Equation (p. 214–216): An equation involving the sine, cosine or tangent functions.
Trigonometric Function (p. 212): The functions sine, cosine and tangent etc.
Turning Point (symbol TP) **(p. 103, 182):** A point where a function changes from increasing to decreasing (or vice versa) and the gradient of a tangent at the point is zero. Nature of, **p. 183**; finding turning points, **p. 184–185**.

Ungrouped (p. 273): Data displayed as single units.
Unlike terms (p. 4): Terms which cannot be combined by adding or subtracting.
Unstructured Questions (p. 261–262): Questions for which the answer does not fall into predetermined categories.
Upper Quartile (p. 284): The value for which 75% of the sample or population is less than or equal to. On a cumulative frequency graph, **p. 296**; on a box plot, **p. 302**.

Variable (p. 4, 37): A letter which represents a number.
Variance (symbol s^2) **(p. 285):** A measure of spread.
Velocity (symbol v) **(p. 187, 241):** The instantaneous rate of change of distance with respect to time.
Venn Diagrams (p. 325): A diagram used to illustrate the relationships between two or more sets.
Vertical Asymptote (p. 114): A vertical line which a graph approaches but does not cross. The value which is excluded from the domain of a hyperbola.
Vertical Line Test (p. 48): Applied to a graph to find if a relation is a function.
Vertices (p. 364): Points in a network.

Word Problems (p. 18, 64, 92): Problems written in words rather than in mathematical form.

***x* Intercepts (p. 48, 74):** The points where a graph cuts the *x* axis.

***y* Intercept (p. 48):** The points where a graph cuts the *y* axis.

APPENDIX

Mathematical Symbols

Sets

$b \in A$	b is an element of A.
$c \notin A$	c is not an element of A.
U or ε	The universal set.
ϕ or { }	The empty or null set.
A'	The complement of A.
$A \subset B$	A is a subset of B.
$A \not\subset B$	A is not a subset of B.
$A \cup B$	The union of A and B, i.e., the set containing elements in A or B.
$A \cap B$	The intersection of A and B, i.e., the set containing elements in both A and B.
n(A)	The number of elements in set A.
$A \leftrightarrow B$	A is equivalent to B, i.e., A and B contain the same number of elements.
$A = B$	A is equal to B if they have exactly the same elements.
$A \times B$	The cross product of A and B.

Number Sets

N	Natural numbers.
W	Whole numbers.
I	Integer numbers.
Q	Rational numbers.
Q′	Irrational numbers.
R	Real numbers.

Language

:	such that
...	and so on
$\therefore$	therefore
$=$	is equal to
$\neq$	is not equal to
$\approx$	is approximately equal to
$<$	is less than
$\leq$	is less than or equal to
$>$	is greater than
$\geq$	is greater than or equal to
$\Rightarrow$	implies
Σ	sum
$\sqrt{x}$	the positive square root of x
$\pm x$	plus or minus x
$\lvert x \rvert$	the absolute value of x

Geometry

$\angle$	angle
$\angle ABC$ or $A\hat{B}C$	The angle defined by the points A, B and C.
$\overline{AB}$	The line segment A to B.
//	parallel to
$\perp$	perpendicular to
Δ	triangle
$^\circ$	degree
π	Pi = 3.14159

Probability

P(A)	The probability of event A occurring.
$\bar{x}$	The mean of a sample.
s.d. or sd	The standard deviation of a sample.

Mathematical Formulae

Area

$\frac{1}{2}bh$	triangle	a^2	square, side a
bh	rectangle or parallelogram	$\frac{1}{2}(a+b)h$	trapezium
$\frac{1}{2}d_1d_2$	rhombus or kite	πr^2	circle
$\frac{\theta}{360} \times \pi r^2$	sector	$4\pi r^2$	sphere

Volume

l^3	cube, side l.	$\frac{1}{3}Ah$	pyramid (A = base area)
lbh	cuboid	$\frac{4}{3}\pi r^3$	sphere
$\frac{1}{2}bhl$	triangular prism	$\frac{1}{3}\pi r^2 h$	cone
$\pi r^2 h$	cylinder		

Logarithms and Indices

Logarithms

If $y = b^x$ then $\log_b y = x$

$\log_b x + \log_b y = \log_b xy$

$\log_b x - \log_b y = \log_b \frac{x}{y}$

$\log_b x^n = n\log_b x$

If $y = e^x$ then $x = ln\, y = \log_e y$

$\log_b x = \frac{\log_a x}{\log_a b}$

$\log_e x = 2.3026 \times \log_{10} x$

Indices

$a^m \times a^n = a^{m+n}$ $\quad a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$ $\quad (ab)^m = a^m b^m$

$a^0 = 1$ (if $a \neq 0$) $\quad \sqrt{a^m} = a^{\frac{m}{2}}$

$\frac{1}{a^m} = a^{-m}$ (if $a \neq 0$)

Co-ordinate Geometry

Distance

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Mid-point of a Line

$$\text{mid-point } \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Lines

$y = mx + c$

$y - y_1 = m(x - x_1)$

Perpendicular and Parallel Lines

$y = m_1 x + c_1$ and $y = m_2 x + c_2$ are parallel if $m_1 = m_2$ and perpendicular if $m_1 \times m_2 = -1$

Circles

$x^2 + y^2 = r^2$ is the circle with centre (0,0) and radius r

$(x-a)^2 + (y-b)^2 = r^2$ circle with centre (a,b) and radius r

Perimeter

$C = \pi d$ or $2\pi r$ circumference of a circle

$\frac{\theta}{360} \times 2\pi r$ arc length of a sector

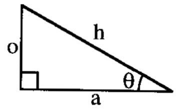
Trigonometry

$\sin\theta = \frac{o}{h}$ o = opposite

$\cos\theta = \frac{a}{h}$ h = hypotenuse

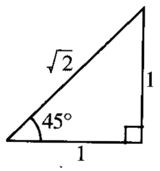
$\tan\theta = \frac{o}{a}$ a = adjacent side

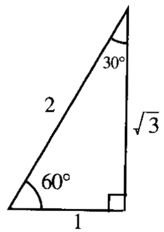
$h^2 = a^2 + o^2$ Pythagoras' theorem



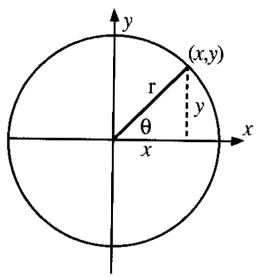
Special Angles

θ	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	





Trigonometric Identities



$\sin(90° \pm \theta) = \cos\theta$

$\cos(90° \pm \theta) = \mp\sin\theta$

$\sin(180° \pm \theta) = \mp\sin\theta$

$\cos(180° \pm \theta) = -\cos\theta$

$\sin(360° - \theta) = -\sin\theta$

$\cos(360° - \theta) = \cos\theta$

Compound Angles

$\sin(A \pm B) = \sin A \times \cos B \pm \cos A \times \sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \times \tan B}$

$\cos(A \pm B) = \cos A \times \cos B \mp \sin A \times \sin B$

Sums and Products

sin C + sin D = 2sin (half sum) × cos (half diff.)

sin C – sin D = 2cos (half sum) × sin (half diff.)

cos C + cos D = 2cos (half sum) × cos (half diff.)

cos D – cos C = 2sin (half sum) × sin (half diff.)

2sin A × cos B = sin (A + B) + sin (A – B)

2cos A × cos B = cos (A + B) + cos (A – B)

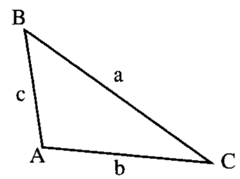
2sin A × sin B = cos (A – B) – cos (A + B)

Triangles

Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Area of triangle: Area = = $\frac{1}{2}ab\sin C$



Sequences and Series

Arithmetic Series

$a + (a+d) + (a+2d) + (a+3d) + \ldots$

$t_n = a + (n-1)d$

$S_n = \frac{n}{2}(2a + (n-1)d)$

Geometric Series

$a + ar + ar^2 + ar^3 + \ldots$

$t_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{(1-r)}, r \neq 1$

$S_\infty = \frac{a}{1-r}$ for $|r| < 1$

Probability

$P(A) = 1 - P(A')$

$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A and B are independent events $\Leftrightarrow P(A \cap B) = P(A) \times P(B)$

Mean, Variance and Standard Deviation

Statistic	Sample of Ungrouped Data	Samples of Grouped Data
Mean	$\bar{x} = \frac{\Sigma x_i}{n}$	$\bar{x} = \frac{\Sigma x_i f_i}{\Sigma f_i}$
Variance	$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$ $= \frac{\Sigma x_i^2}{n} - \bar{x}^2$	$s^2 = \frac{\Sigma(x - \bar{x})^2 f_i}{\Sigma f_i}$ $= \frac{\Sigma x_i^2 f_i}{\Sigma f_i} - \bar{x}^2$
Standard Deviation	$s = \sqrt{s^2}$	$s = \sqrt{s^2}$

Calculus

Differentiation of a Sum

$(f + g)' = f' + g'$

Product Rule

$(f \times g)' = f' \times g + f \times g'$

i.e. $\frac{d(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

Quotient Rule

$\left(\frac{f}{g}\right)' = \frac{g \times f' - f \times g'}{g^2}$

i.e. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Composite Function Rule

$(fog)' = f'og \times g'$

i.e. if $u = g(x)$ and $y = f(u)$ then

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Integration

Function $y = f(x)$	Indefinite Integral $\int y dx = \int f(x) dx$	Definite Integral $\int_a^b y dx = \int_a^b f(x) dx$
0	k	0
c	$cx + k$	$c(b - a)$
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + k$	$\frac{b^{n+1} - a^{n+1}}{n+1}$

Areas Under Normal Probability Curve

The tabulate value is the probability that the standardised normal variate z (with $\mu=0$, $\sigma=1$) lies between 0 and z e.g. $P(0 < z < 1.43) = 42.36\%$

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0275	.0319	.0359	4	8	12	16	20	24	28	32	36
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754	4	8	12	16	20	24	28	32	36
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141	4	8	12	15	19	22	27	31	35
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517	4	8	11	15	19	22	26	30	34
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879	4	7	11	14	18	22	25	29	32
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224	3	7	10	14	17	21	24	27	31
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549	3	6	10	13	16	19	23	26	29
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852	3	6	9	12	15	18	21	24	27
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133	3	6	8	11	14	17	19	22	25
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389	3	5	8	10	13	15	18	20	23
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621	2	5	7	9	12	14	16	18	21
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830	2	4	6	8	10	12	14	16	19
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015	2	4	5	7	9	11	13	15	16
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177	2	3	5	6	8	10	11	13	14
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319	1	3	4	6	7	8	10	11	13
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441	1	2	4	5	6	7	8	10	11
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545	1	2	3	4	5	6	7	8	9
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633	1	2	3	3	4	5	6	7	8
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4669	.4706	1	1	2	3	4	4	5	6	6
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767	1	1	2	2	3	4	4	5	5
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817	0	1	1	2	2	3	3	4	4
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857	0	1	1	2	2	2	3	3	4
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890	0	1	1	1	2	2	2	3	3
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916	0	0	1	1	1	2	2	2	2
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936	0	0	1	1	1	1	1	2	2
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952	0	0	0	1	1	1	1	1	1
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964	0	0	0	0	1	1	1	1	1
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974	0	0	0	0	0	1	1	1	1
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981	0	0	0	0	0	0	0	0	1
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986	0	0	0	0	0	0	0	0	1
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990	0	0	0	0	0	0	0	0	0
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993	0	0	0	0	0	0	0	0	0
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995	0	0	0	0	0	0	0	0	0
3.3	.4995	.4995	4.995	.4996	.4996	.4996	.4996	.4996	.4996	.4997	0	0	0	0	0	0	0	0	0
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998	0	0	0	0	0	0	0	0	0
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	0	0	0	0	0	0	0	0	0
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	0	0	0	0	0	0	0	0	0
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	0	0	0	0	0	0	0	0	0

Government of Samoa
MINISTRY OF EDUCATION, SPORTS & CULTURE